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Symmetry and enumerative geometry of Calabi–Yau spaces

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Abstract

This thesis contains two research papers. The main objects of interest are moduli spaces of sheaves and curves on Calabi–Yau spaces, e.g. K3 surfaces and Calabi–Yau 3-folds.

In the first paper, in joint work with Miguel Moreira, we study the curve counting invariants of Calabi–Yau 3-folds via the Weyl reflection along a ruled divisor. We obtain a new rationality result and functional equation for the generating functions of Pandharipande–Thomas invariants. When the divisor arises as resolution of a curve of A_1-singularities, our results match the rationality of the associated Calabi–Yau orbifold.

The symmetry on generating functions descends from the action of an infinite dihedral group of derived auto-equivalences, which is generated by the derived dual and a spherical twist. Our techniques involve wall-crossing formulas and generalized DT invariants for surface-like objects.

In the second paper, in joint work with Younghan Bae, we prove a conjecture of Maulik, Pandharipande, and Thomas expressing the Gromov–Witten invariants of K3 surfaces for divisibility two curve classes in all genus in terms of weakly holomorphic quasimodular forms of level two. Then, we establish the holomorphic anomaly equation in divisibility two in all genus.

Our approach involves a refined boundary induction, relying on the top tautological group of the moduli space of smooth curves, together with a degeneration formula for the reduced virtual fundamental class with imprimitive curve classes. We use the double ramification relations with target variety as a new tool to prove the initial condition. The relationship between the holomorphic anomaly equation for higher divisibility and the conjectural multiple cover formula of Oberdieck and Pandharipande is discussed in detail and illustrated with several examples.

Zusammenfassung

Die vorliegende Doktorarbeit besteht aus zwei Artikeln. Von zentraler Bedeutung sind Modulräume von Garben und Kurven in Calabi–Yau Räumen, z.B. K3 Flächen und Calabi–Yau 3-Faltigkeiten.

Im ersten Artikel, in gemeinsamer Arbeit mit Miguel Moreira, studieren wir die enumerativen Invarianten von Calabi–Yau 3-Faltigkeiten mittels der Weyl Reflektion entlang eines geregelten Divisors. Wir erhalten neue Resultate zur Rationalität und Funktionalgleichungen für die erzeugenden Funktionen von Pandharipande–Thomas Invarianten. Wenn der Divisor als Auflösung einer Kurve von A_1-Singularitäten auftritt, stimmen unsere Ergebnisse mit der Rationalität der assoziierten Calabi–Yau Orbifaltigkeit überein.

Die Symmetrie der erzeugenden Funktionen beruht auf der Wirkung einer unendlichen Diedergruppe von derivierten Autoäquivalenzen, welche vom dualisierenden Funktor und einem sphärischen Twist erzeugt wird. Unsere Techniken umfassen Wall-Crossing Formeln und verallgemeinerte DT Invarianten für Flächenähnliche Objekte.

Im zweiten Artikel, in gemeinsamer Arbeit mit Younghan Bae, beweisen wir eine Vermutung von Maulik, Pandharipande und Thomas, welche die Gromov–Witten Invarianten von K3 Flächen für Kurvenklassen von Divisibilität zwei und beliebigem Geschlecht mittels schwach holomorpher Quasimodulformen vom Level zwei ausdrückt. Anschliessend zeigen wir die holomorphe Anomalie Gleichung in Divisibilität zwei für beliebiges Geschlecht.

Unser Ansatz verwendet ein verfeinertes Induktionsverfahren, beruhend auf der Struktur der höchsten tautologischen Gruppe des Modulraums glatter Kurven, zusammen mit einer Degenerationsformel für die reduzierte virtuelle Fundamentalklasse für imprimitive Kurvenklassen. Wir verwenden die verallgemeinerten Relationen von Zykeln doppelter Ramifizierung als neues Hilfsmittel um gewisse Anfangsbedingungen zu beweisen. Wir erklären die Beziehung zwischen der holomorphen Anomalie Gleichung für höhere Divisibilität und die vermutete Formel von Oberdieck und Pandharipande im Detail und wir geben mehrere Beispiele zur Illustration.

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Chapter 1
Introduction

This thesis contains two research papers [3, 11]. The main objects of interest are moduli spaces of sheaves and curves on Calabi–Yau spaces, e.g. K3 surfaces and Calabi–Yau 3-folds. Their invariants (GW/DT/PT, motivic) are connected to string theory and are of major interest to mathematicians and physicists. Computations are notoriously difficult, but recent progress has been spectacular and new, powerful techniques for effective calculations were introduced [10, 18, 23, 24, 26, 29].

For a Calabi–Yau 3-fold X the goal is to determine the *partition function* Z^X, which gathers all numerical data in a single function. The famous MNOP conjecture[1] [19, 20] asserts the equivalence of Gromov–Witten theory and Donaldson–Thomas theory. Either curve counting theory yields the same function Z^X. We use both theories to study Z^X.

The GW theory has strong properties such as modularity and holomorphic anomaly equations [6, 7]. The DT theory allows to unveil hidden symmetries related to the derived category. For $K3 \times E$ (smooth elliptic curve E) this strategy was beautifully executed [25, 27]:

$$Z^{K3\times E} = -\frac{1}{\chi_{10}}\,.$$

The function is a meromorphic Siegel modular form. My work focuses on the strict Calabi–Yau 3-fold case ($H^1(\mathcal{O}_X) = 0$). A modular solution is expected, no proposal exists. Below I describe our techniques for progress, which were developed in two papers, one for each curve counting theory.

In the first paper (joint with M. Moreira) [11] we developed a framework to use *spherical twists*, certain derived autoequivalences of Calabi–Yau 3-folds, to deduce constraints on Pandharipande–Thomas stable pairs invariants. We provide rigorous mathematical proofs of predictions made by physicists [15, 16, 17] in the early 2000's.

[1] The conjecture has been proved in many cases [21, 28].

In the second paper (joint with Y. Bae) [3] we leveraged new results on *universal double ramification cycles* [2] to prove a conjecture of Maulik, Pandharipande, and Thomas [22] on GW invariants of K3 surfaces. The results concern the multiple cover formula and the holomorphic anomaly equation.

1 Weyl symmetry and spherical twists

Let X be a smooth projective Calabi–Yau 3-fold. The Pandharipande–Thomas invariants $\mathrm{PT}_{\beta,n} \in \mathbb{Z}$ are curve counting invariants enumerating stable pairs in the derived category $D^b(X)$ of curve class $\beta \in H_2(X,\mathbb{Z})$ and Euler characteristic $n \in \mathbb{Z}$. This theory is equivalent to the more classical Donaldson–Thomas theory counting ideal sheaves of embedded curves. The associated generating function of invariants is the partition function Z^X. We study the symmetries of Z^X induced by the action of the group of derived autoequivalences

$$\mathrm{Aut}\left(D^b(X)\right).$$

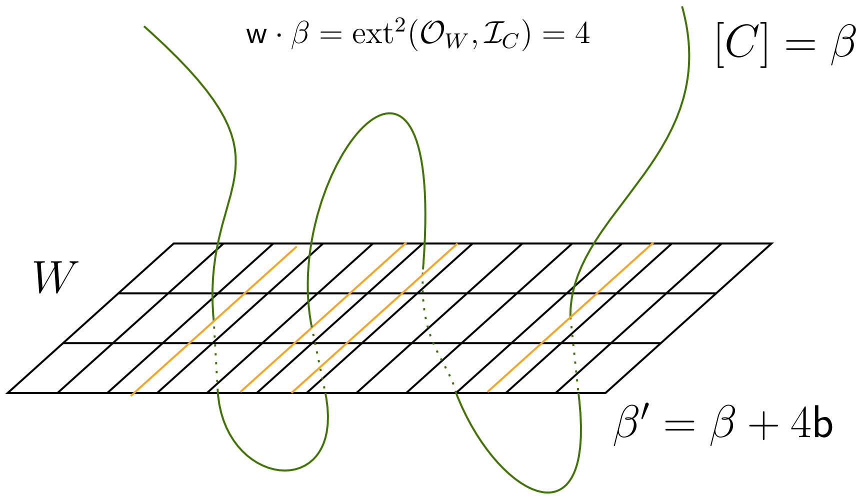
The modern way to obtain constraints on Z^X is to use motivic Hall algebras, Joyce's generalized DT invariants, and wall-crossing formula [13, 14]. We were able to succesfully employ this strategy in [11]. The derived autoequivalence is directly related to the geometry of X.

1.1 Geometry

Let X be a Calabi–Yau 3-fold containing a smooth geometrically ruled divisor W. Physical considerations for BPS state counts [15, 17] suggest that the curve counting invariants of X are constrained by this constellation. More precisely, let $\mathsf{w} \in H_4(X,\mathbb{Z})$ be the class of the divisor and $\mathsf{b} \in H_2(X,\mathbb{Z})$ be the class of the rational curve of the ruling. Consider the *Weyl symmetry* on $H_2(X,\mathbb{Z})$ defined by

$$\beta \longmapsto \beta' = \beta + (\mathsf{w} \cdot \beta)\, \mathsf{b}\,.$$

Since $\mathsf{w} \cdot \mathsf{b} = -2$, this defines a reflection. For a smooth curve C representing the class β we have the following illustration:



The intersection product $\mathsf{w} \cdot \beta$ agrees with the dimension of a certain Ext-group. This fact is reminiscent of Serre's intersection formula. Building on this observation we are able to lift the Weyl symmetry to a derived autoequivalence

$$\rho \in \mathrm{Aut}\left(D^b(X)\right)$$

and consider the action on the generating series of stable pairs invariants

$$\mathrm{PT}_\beta(q, Q) = \sum_{n, j \in \mathbb{Z}} \mathrm{PT}_{\beta + j\mathsf{b}, n} \, (-q)^n \, Q^j \, .$$

1.2 Main result

Under mild assumptions on the geometry we prove the following result.

Theorem 1.1 ([11, Theorem 1.1])**.**

$$\frac{\mathrm{PT}_\beta(q, Q)}{\mathrm{PT}_0(q, Q)} \in \mathbb{Q}(q, Q)$$

is the expansion of a rational function $f_\beta(q, Q)$ with functional equations

$$f_\beta(q^{-1}, Q) = f_\beta(q, Q) \, ,$$
$$f_\beta(q, Q^{-1}) = Q^{-\mathsf{w} \cdot \beta} \, f_\beta(q, Q) \, .$$

The first functional equation is the $q \leftrightarrow q^{-1}$ invariance induced by the derived dual [8, 31, 32]. It is of fundamental importance for the MNOP correspondence.

The second functional equation is induced by ρ and precisely corresponds to the Weyl symmetry. A basic example of such a rational function is (related to the local geometry $K_{\mathbb{P}^1\times\mathbb{P}^1}$)

$$f(q,Q) = \frac{q^2}{(1-qQ)^2(q-Q)^2}\,.$$

The derived autoequivalence ρ arises as a composition of the derived dual and a *spherical twist* [1, 12, 30]. The induced action on stable pairs leads to torsion-free objects in a certain heart of bounded t-structure in the derived category and we use a two-dimensional family of stability conditions to interpolate between the two. The proof involves a careful analysis of the wall-crossing behavior and the combinatorics of the wall-crossing formulas closely related to [5].

2 Gromov–Witten invariants of K3 surfaces

The enumerative geometry of smooth complex projective K3 surfaces concerns questions of classical nature such as the number of rational curves in a given linear system, as well as questions in Gromov–Witten theory or Donaldson–Thomas theory. In the last two decades the field has seen substantial development in all of these areas, and the interplay between them has led to proofs of longstanding conjectures.

2.1 Modularity

The enumerative invariants associated to K3 surfaces exhibit connections to number theory in remarkable ways. The first instance of this is the following beautiful formula conjectured by Yau–Zaslow [33] and proved in [4, 9]

$$\sum_{h\geq 0} N_h\, q^{h-1} = q^{-1}\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^{24}}\,. \tag{1.1}$$

Here, the numbers N_h are the counts of rational curves in a given primitive linear series of arithmetic genus h. The right hand side of (1.1) is the inverse of the famous discriminant modular form. This theme generalizes to the Gromov–Witten theory of K3 surfaces: gathered as generating series, these invariants are expected to form meromorphic quasimodular forms in all cases. In a seminal work Maulik, Pandharipande, and Thomas [22] provided proofs of this statement for primitive curve clases, and introduced powerful techniques to calculate the invariants. A precise conjecture about the modularity was put forward. In a joint work with Y. Bae we were able to prove this conjecture for *all descendents* and *all genus* when the curve class has divisibility ≤ 2. We can also deduce the *holomorphic anomaly equation*, a differential equation expressing the formal derivative of a quasimodular form with respect to the Eisenstein series C_2 in terms of lower genus data.

Theorem 2.1 ([3, Thm. 1])**.** For all homogeneous $\gamma_1, \ldots, \gamma_n \in H^*(S)$ the generating series of descendent GW invariants

$$\left\langle \tau_{a_1}(\gamma_1), \ldots, \tau_{a_n}(\gamma_n) \right\rangle_{g,2} \in \frac{1}{\Delta(q)^2}\mathsf{QMod}(2)$$

is the expansion of a level 2 quasimodular form. Moreover, the formal C_2-derivative satisfies the holomorphic anomaly equation.

Of special interest here is the divisibility of the curve class. For primitive classes, an algorithm [22] involving degeneration, localization, and strong results about the tautological ring $R^*(\overline{M}_{g,n})$ allows calculation of the Gromov–Witten invariants. The procedure ultimately reduces to the Yau–Zaslow formula (1.1), which lies at the heart of the modularity. This algorithm breaks down for imprimitive curve classes. The basic reason is that the procedure reduces to a much more complicated, although finite, set of initial conditions. In our work, we are able to provide proofs for these initial conditions. The key ingredient are the *double ramification relations* as developed in [2], which *do not depend on the divisibility* of the curve class. Special properties of the reduced obstruction theory (the splitting behavior of reduced classes) for K3 targets are essential.

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Chapter 2

Weyl symmetry for curve counting invariants via spherical twists

Tim-Henrik Buelles and Miguel Moreira

1 Introduction

1.1 Overview

Let X be a Calabi–Yau 3-fold containing a smooth geometrically ruled divisor W. Physical considerations for BPS state counts [24, 27] suggest that the curve counting invariants of X are constrained by this constellation. More precisely, let $\mathsf{w} \in H_4(X,\mathbb{Z})$ be the class of the divisor and $\mathsf{b} \in H_2(X,\mathbb{Z})$ be the class of the rational curve of the ruling. Consider the *Weyl symmetry* on $H_2(X,\mathbb{Z})$ defined by

$$\beta \longmapsto \beta' = \beta + (\mathsf{w}\cdot\beta)\,\mathsf{b}\,.$$

Since $\mathsf{w}\cdot\mathsf{b} = -2$, this defines a reflection. The guiding example is that of an elliptic Calabi–Yau 3-fold

$$X \to \mathbb{P}^1 \times \mathbb{P}^1$$

which is fibered in elliptic K3 surfaces over $\mathbb{P}^1$, see Section 1.6. For K3 curve classes β, the Weyl symmetry $\beta \leftrightarrow \beta'$ is exactly realized on the level of Gopakumar–Vafa invariants

$$n^{K3}_{g,\beta} = n^{K3}_{g,\beta'}\,.$$

The equality is reminiscent of the monodromy for quasi-polarized K3 surfaces. For arbitrary $\beta \in H_2(X,\mathbb{Z})$ such an equality cannot hold, for example when $\beta \in H_2(W,\mathbb{Z})$ in which case the invariants are given by the local surface K_W. Instead, we find that the Weyl symmetry *is realized as a functional equation.* This symmetry is analogous to the rationality and the $q \leftrightarrow q^{-1}$ invariance for generating series of Pandharipande–Thomas stable pairs invariants.

The Pandharipande–Thomas (PT) [40] invariants $\mathrm{PT}_{\beta,n} \in \mathbb{Z}$ are curve counting invariants enumerating stable pairs in the derived category $D^b(X)$ with curve class

β and Euler characteristic $n \in \mathbb{Z}$. Our results concern the 2-variable generating series[1]

$$\mathrm{PT}_\beta(q,Q) = \sum_{n,j\in\mathbb{Z}} \mathrm{PT}_{\beta+j\mathsf{b},n}\,(-q)^n\,Q^j\,.$$

Let $\mathcal{E}$ be a rank 2 bundle over a smooth projective curve C of genus g and $p\colon W = \mathbb{P}_C(\mathcal{E}) \to C$ be the corresponding $\mathbb{P}^1$-bundle. We will assume that X admits a nef class $A \in \mathrm{Nef}(X)$ which vanishes only on the extremal ray spanned by b, i.e.[2]

$$\mathrm{Ker}\left(N_1^{\mathrm{eff}}(X) \xrightarrow{A\cdot} \mathbb{Z}\right) = \mathbb{Z}_{\geq 0}\cdot\mathsf{b}\,. \tag{$\diamondsuit$}$$

The generating series PT_0 of multiples of b is easily computed as

$$\mathrm{PT}_0(q,Q) = \prod_{j\geq 1}(1-q^jQ)^{(2g-2)j}.$$

Our main result is:

Theorem 1.1. Let X be a Calabi–Yau 3-fold containing a smooth divisor W satisfying condition ($\diamondsuit$). Then

$$\frac{\mathrm{PT}_\beta(q,Q)}{\mathrm{PT}_0(q,Q)} \in \mathbb{Q}(q,Q)$$

is the expansion of a rational function $f_\beta(q,Q)$ such that

$$\begin{aligned} f_\beta(q^{-1},Q) &= f_\beta(q,Q)\,,\\ f_\beta(q,Q^{-1}) &= Q^{-\mathsf{w}\cdot\beta}\,f_\beta(q,Q)\,. \end{aligned}$$

The rationality in q and the invariance under $q \leftrightarrow q^{-1}$ are well-known [8, 40, 47, 50]. The symmetry is induced by the action of the derived dual $\mathbb{D}^X$ on $D^b(X)$. Analogously, we introduce a derived anti-equivalence ρ of order two, which promotes the Weyl reflection to the derived category and induces the second functional equation on generating functions. It is defined as

$$\rho = t_\Phi \circ \mathbb{D}^X\,,$$

where t_Φ is a derived equivalence of infinite order induced by a spherical functor Φ.

The image of a stable pair under ρ leads to complicated objects in the derived category and a symmetry on invariants is not easily deduced. Instead, we consider an abelian category

$$\mathcal{A} \subset D^{[-1,0]}(X)\,,$$

[1] We use the non-standard sign $-q$ which simplifies some formulas.

[2] We do not require the line bundle to be basepoint-free and we do not assume a contraction morphism. Such a nef class exists in many cases, e.g. for elliptic Calabi–Yau 3-folds.

defined as a tilt of $\mathrm{Coh}(X)$ along a torsion pair. The action of ρ on $\mathcal{A}$ is analogous to the action of $\mathbb{D}^X$ on $\mathrm{Coh}(X)$. In particular, we consider a notion of dimension which is preserved by ρ (up to shift). Define the extension closure

$$ {}^p\mathcal{B} = \left\langle \mathcal{O}_X[1]\,, \mathcal{A}_{\leq 1} \right\rangle . $$

The action of ρ induces a symmetry for *perverse PT invariants* ${}^p\mathrm{PT}_{\gamma,n}$ enumerating torsion-free objects in ${}^p\mathcal{B}$. These objects are allowed to have non-trivial first Chern class a multiple of the class w. For $r \in \mathbb{Z}$ and $\gamma = (r\mathsf{w}, \beta)$ define the generating series

$$ {}^p\mathrm{PT}_\gamma(q, Q) = \sum_{n,j\in\mathbb{Z}} {}^p\mathrm{PT}_{\gamma+j\mathsf{b},n}\,(-q)^n\,Q^j \in \mathbb{Q}[[q^{\pm 1}, Q^{\pm 1}]]\,. $$

The rationality and functional equation for ${}^p\mathrm{PT}_\gamma$ is proved via Joyce's wall-crossing formula [22]. The formula involves generalized DT invariants for surface-like objects supported on W with non-trivial Euler pairings.

Theorem 1.2.

$$ {}^p\mathrm{PT}_\gamma(q, Q) \in \mathbb{Q}(q, Q) $$

is the expansion of a rational function $f_\gamma \in \mathbb{Q}(q, Q)$ with functional equation

$$ f_\gamma(q^{-1}, Q^{-1}) = Q^{-\mathsf{w}\cdot\beta+(2-2g)r}\, f_\gamma(q, Q)\,. $$

Theorem 1.1 is a consequence of Theorem 1.2 in the special case $r = 0$, together with the $q \leftrightarrow q^{-1}$ symmetry. The comparison between stable pairs and perverse stable pairs is given by a second wall-crossing. The following formula holds as an equality of rational functions but not necessarily as generating series.

Theorem 1.3.

$$ {}^p\mathrm{PT}_{(0,\beta)}(q, Q) = \frac{\mathrm{PT}_\beta(q, Q)}{\mathrm{PT}_0(q, Q)}\,. $$

1.2 Crepant resolution

The results and techniques of this paper are strongly influenced by the recent proof of the crepant resolution conjecture by Beentjes, Calabrese and Rennemo [5] for Donaldson–Thomas (DT) invariants [44]. Consider a type III contraction $X \to Y$ with exceptional divisor W, contracting the rational curves of the ruling. Assume that $X \to Y$ is the distinguished crepant resolution of the (singular) coarse moduli space of a Calabi–Yau orbifold $\mathcal{Y}$

$$ \begin{array}{ccc} \mathcal{Y} & & X \\ & \searrow \quad \swarrow & \\ & Y & \end{array} $$

The derived McKay correspondence proposed by Bridgeland, King, and Reid [9] induces a derived equivalence

$$\Phi\colon D^b(X) \xrightarrow{\sim} D^b(\mathcal{Y}),$$

which restricts to an equivalence [13, Theorem 1.4]

$$\mathcal{A} \xrightarrow{\sim} \mathrm{Coh}(\mathcal{Y}).$$

The notion of perverse stable pairs on X coincides with the image of stable pairs on $\mathcal{Y}$. The results of Theorems 1.1, 1.2, and 1.3 are the rationality and functional equation of $\mathrm{PT}(\mathcal{Y})$ and the wall-crossing between $\Phi^{-1}\big(\mathrm{PT}(\mathcal{Y})\big)$ and Bryan–Steinberg pairs of $X \to Y$ [5, 11, 13]. The nef class is given by the pullback of an ample class on Y and the derived anti-equivalence ρ corresponds to the derived dual of $\mathcal{Y}$

$$\rho = \Phi^{-1} \circ \mathbb{D}^{\mathcal{Y}} \circ \Phi.$$

1.3 Spherical twist

Define the functor $\Phi\colon D^b(C) \to D^b(X)$ as

$$\Phi(V) = \iota_*\big(\mathcal{O}_p(-1) \otimes p^*V\big).$$

This defines a spherical functor [2, 17, 42]. Let Φ_R be the right adjoint. The cone of the counit morphism defines the spherical twist t_Φ, an autoequivalence of $D^b(X)$, via

$$\Phi \circ \Phi_R \to \mathrm{id} \to \mathrm{t}_\Phi.$$

The derived dual $\mathbb{D}^X$ and the spherical twist t_Φ generate an infinite dihedral group (containing ρ) which underlies the functional equations of Theorem 1.1.

1.4 Gromov–Witten/ BPS invariants

The second functional equation of Theorem 1.1 implies strong constraints for the enumerative invariants in curve classes $\beta + j\mathsf{b}$ for varying $j \in \mathbb{Z}$ and fixed genus. In particular, finitely many j determine the full set of these invariants. Let $\mathrm{GW}_{h,\beta}$ be the Gromov–Witten invariants of X and assume that the GW/PT correspondence [33, 34, 35, 39] holds for X.

Corollary 1.4. For all $(h,\beta) \neq (0, m\mathsf{b}), (1, m\mathsf{b})$ the series

$$\sum_{j\in\mathbb{Z}} \mathrm{GW}_{h,\beta+j\mathsf{b}}\, Q^j$$

is the expansion of a rational function $f_{h,\beta}(Q)$ with functional equation

$$f_{h,\beta}(Q^{-1}) = Q^{-\mathsf{w}\cdot\beta} f_{h,\beta}(Q).$$

The rational function is expected to have the particular form

$$f_{h,\beta} = \frac{p_{h,\beta}(Q)}{(1-Q)^d}$$

which leads to polynomiality of $\mathrm{GW}_{h,\beta+j\mathsf{b}}$ and the limit behavior of BPS counts (as $j \to \infty$) discussed in the physics literature [23, Section 5]. For the local Hirzebruch surface K_W we give full proofs in Appendix 8.

1.5 Elliptic Calabi–Yau 3-folds

Let $p\colon S \to C$ be a $\mathbb{P}^1$-bundle over a smooth projective curve C and

$$f\colon X \to S$$

an elliptic fibration[3] with a section W. Let D be a sufficiently ample divisor on C such that $-K_S + p^*D$ is ample. A nef class satisfying the condition ($\diamondsuit$) is given by

$$\mathsf{w} + f^*(-K_S + p^*D) \in H^2(X, \mathbb{Z})\,.$$

For any $\beta \in H_2(W, \mathbb{Z})$ define

$$P_\beta(q,t) = \sum_{d \geq 0} \sum_{n \in \mathbb{Z}} \mathrm{PT}_{\beta + d\mathsf{f},n}\, (-q)^n\, t^d\,,$$

where $\mathsf{f} \in H_2(X, \mathbb{Z})$ is the class of a smooth elliptic fiber. Recent considerations in topological string theory [18] predict that

$$Z_\beta(q,t) = \frac{P_\beta(q,t)}{P_0(q,t)}$$

is the expansion of a meromorphic Jacobi form. Theorem 1.1 implies non-trivial constraints among the Jacobi forms $\{Z_{\beta+j\mathsf{b}}\}_{j\in\mathbb{Z}}$.

1.6 STU

Theorem 1.1 and Corollary 1.4 provide mathematical proofs of a heterotic mirror symmetry on BPS invariants as observed in [26]. The symmetry is discussed for type IIA duals of the STU model, i.e. the elliptic Calabi–Yau 3-fold

$$X \to \mathbb{P}^1 \times \mathbb{P}^1$$

[3] Since X is Calabi–Yau, C is necessarily rational and S is a Hirzebruch surface.

such that both projections to $\mathbb{P}^1$ define K3-fibrations with 528 singular fibers with exactly one double point as singularity. This geometry can be constructed as a hypersurface in a toric variety [25].

The symmetry on BPS invariants [26, Section 6.10.3] is realized by the second functional equation of Theorem 1.1 and we can identify the infinite order symmetry [26, Equation 6.65] with the action of t_Φ. The rationality and functional equation of Corollary 1.4 verifies [26, Equation 6.67]. We obtain the precise form of the rational function for the local case $K_{\mathbb{P}^1\times\mathbb{P}^1}$ in Appendix 8.

As a special case of the rationality and functional equation, consider $\beta = h\mathsf{f}$ a multiple of the elliptic fiber class. Then, the generating function is in fact a *Laurent polynomial* in Q and the functional equation

$$f_\beta(q, Q^{-1}) = Q^{-\mathsf{w}\cdot\beta} f_\beta(q, Q)$$

holds at the level of coefficients and recovers the symmetry

$$n^{K3}_{g,m\mathsf{b}+h\mathsf{f}} = n^{K3}_{g,(h-m)\mathsf{b}+h\mathsf{f}}$$

of BPS invariants for K3 surfaces. This symmetry is usually seen as a consequence of the monodromy for quasi-polarized K3 surfaces.

A related geometry, also called an STU model in the physics literature, may be useful towards a crepant resolution conjecture in the *non hard Lefschetz* case. We consider

$$X \to \mathbb{F}_1$$

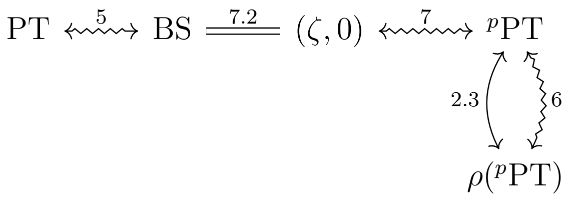
an elliptic Calabi–Yau 3-fold over the Hirzebruch surface $\mathbb{F}_1$. The fibration has a section W and we obtain $X \dashrightarrow X'$ as the Atiyah flop along the rational curve in W of self-intersection -1. After this transformation we have a type II contraction $X' \to Y'$ with exceptional divisor $\mathbb{P}^2$, which is the crepant resolution of an isolated canonical singularity. After the flop formula for DT invariants [12, 48], the symmetry of Theorem 1.1 must induce a symmetry on X'.

1.7 Outline

We briefly sketch the strategy of the paper. Section 2 contains a discussion of perverse sheaves and the perverse t-structure associated to the geometry. We introduce the anti-equivalence ρ and show several important properties that will be needed in the later parts. In Section 3 we recall some facts about Hall algebras, pairs, and wall-crossing, and we set notation for the rest of the paper. Stability conditions play an important role for this paper and we comment on them in Section 4. In Section 5 we introduce invariants which resemble Bryan–Steinberg invariants [11] and we prove a wall-crossing formula between those and usual PT invariants. The wall-crossing formula shows a relation of the form

$$\mathrm{BS}_\beta = \frac{\mathrm{PT}_\beta}{\mathrm{PT}_0}$$

and thus gives a natural interpretation to the quotient on the right hand side. The rationality and symmetry for ${}^p\mathrm{PT}$ invariants are proven in Section 6. Essentially, the result is obtained by comparing ${}^p\mathrm{PT}$ invariants with $\rho({}^p\mathrm{PT})$ invariants in two ways: first using the anti-equivalence ρ, and then using wall-crossing. In Section 7 we describe a wall-crossing between the BS invariants and the perverse ${}^p\mathrm{PT}$ invariants (which in the crepant case $X \to Y$ are the orbifold invariants). An important aspect is that while PT and BS invariants are defined using the integration map on the Hall algebra obtained from the heart $\mathrm{Coh}(X) \subset D^b(X)$, the perverse ${}^p\mathrm{PT}$ invariants are defined using the heart $\mathcal{A} \subset D^b(X)$. The ζ-wall-crossing of Section 7 takes place in $\mathcal{A}$. In Section 7.2 we identify BS-pairs as the pairs in the end of the ζ–wall-crossing. The following diagram represents the different invariants we use in the paper and their relations. The squiggly arrows represent wall-crossing formulas.



1.8 Related work

The following question was posed by Toda [49]:

Question. How are stable pair invariants on a Calabi–Yau 3–fold constrained, due to the presence of non-trivial autoequivalences of the derived category?

The most famous instance is the rationality and functional equation induced by the derived dual. Similarly, the elliptic transformation law for $Z_\beta(q,t)$ is deduced from a derived involution [38]. Significant progress for abelian 3-folds was made using Bridgeland stability conditions [37]. The Seidel–Thomas spherical twist for an embedded $\mathbb{P}^2$ was considered in [49] and certain polynomial relations for stable pairs invariants were obtained. Our results provide an answer to Question 1.8 for the involution ρ. The flop construction $X \dashrightarrow X'$ of the previous section must connect our results with the ones obtained in [49, Theorem 1.2].

1.9 Conventions

We work over the complex numbers. The canonical bundle of W is denoted K_W. Intersection products are denoted by a dot, e.g. $\mathsf{w} \cdot \beta$. Stable pairs are considered in cohomological degree -1 and 0. This convention follows [5] and differs from [40].

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2 Perverse t-structure

In this section we give a self-contained discussion of perverse sheaves and duality associated to the following geometry.

2.1 Geometry

Let C be a smooth projective curve, $\mathcal{E}$ a locally free sheaf of rank 2, and $W = \mathbb{P}_C(\mathcal{E})$ a geometrically ruled surface with projection $p\colon W \to C$. We assume that $\mathcal{E}^\vee$ is globally generated[4] and we fix line bundles $L_1, L_2 \in \mathrm{Pic}(C)$ such that

$$0 \to L_1 \to \mathcal{E}^\vee \to L_2 \to 0\,.$$

Let X be a smooth projective Calabi–Yau 3-fold containing W as a divisor:

$$\begin{array}{ccc} W & \overset{\iota}{\hookrightarrow} & X \\ {\scriptstyle p}\big\downarrow & & \\ C & & \end{array}$$

The curve class of a fiber of p (and its pushforward to X) is denoted b. The nef class A of condition $(\diamondsuit)$ restricts to a multiple of the fiber class, i.e. $\iota^* A$ is

[4] Twisting $\mathcal{E}^\vee$ with an ample line bundle does not change the geometry $\mathbb{P}_C(\mathcal{E})$.

numerically equivalent to $a_0\mathsf{b}$ for some $a_0 \in \mathbb{Z}_{>0}$. Recall that we have the Euler sequence on W which we will use repeatedly:

$$0 \to \omega_p \to \mathcal{O}_p(-1) \otimes p^*\mathcal{E}^\vee \to \mathcal{O}_W \to 0\,.$$

2.2 Torsion pair

Define the category

$$\mathcal{T} = \left\{ T \in \mathrm{Coh}(X) \mid R^1p_*(\iota^*T) = 0 \right\}.$$

Lemma 2.1. The subcategory $\mathcal{T} \subset \mathrm{Coh}(X)$ is closed under extensions and quotients in $\mathrm{Coh}(X)$.

Proof. Use the long exact sequence of higher pushforward sheaves and the fact that $R^2p_* = 0$ since the fibers of p are 1-dimensional. □

Define the orthogonal complement

$$\mathcal{F} = \mathcal{T}^\perp = \{ F \in \mathrm{Coh}(X) \mid \mathrm{Hom}(T, F) = 0 \text{ for all } T \in \mathcal{T} \}\,.$$

By [48, Lemma 2.15] we obtain a torsion pair $(\mathcal{T}, \mathcal{F})$ of $\mathrm{Coh}(X)$.

Throughout, if a sequence of full subcategories $\mathcal{C}_1, \ldots, \mathcal{C}_m \subset \mathcal{C}$ of some abelian category $\mathcal{C}$ forms a torsion m-tuple [50, Definition 3.6] we write

$$\mathcal{C} = \left\langle \mathcal{C}_1, \ldots, \mathcal{C}_m \right\rangle.$$

We consider the *perverse t-structure* on $D^b(X)$ whose heart is the tilt [16]

$$\mathcal{A} = \left\langle \mathcal{F}[1], \mathcal{T} \right\rangle.$$

Every $E \in D^b(X)$ has associated perverse cohomology ${}^p\mathcal{H}^i(E) \in \mathcal{A}$ and exact triangles lead to long exact sequences of perverse cohomology. Define the *perverse dimension*

$${}^p\dim(E) = \max\left\{ \dim \mathrm{supp}(E) \cap (X \setminus W)\,, \dim p\big(\mathrm{supp}(E) \cap W\big) \right\}.$$

We write $\mathcal{A}_{\le k}$ for elements of $\mathcal{A}$ with perverse dimension at most k and $\mathcal{A}_k$ for elements with pure perverse dimension k, i.e.

$$\mathrm{Hom}(\mathcal{A}_{\le k-1}, \mathcal{A}_k) = 0\,.$$

We also denote $\mathcal{F}_k[1] = \mathcal{F}[1] \cap \mathcal{A}_k$ and $\mathcal{T}_k = \mathcal{T} \cap \mathcal{A}_k$.

2.3 Duality

The derived dualizing functor $(-)^\vee = R\mathcal{H}om(-, \mathcal{O}_X)$ is a duality for the standard t-structure on $D^b(X)$. We introduce a duality ρ on $D^b(X)$ which is the analog for the perverse t-structure.

Define the functor $\Phi\colon D^b(C) \to D^b(X)$ as

$$\Phi(V) = \iota_*\big(\mathcal{O}_p(-1) \otimes p^*V\big)\,.$$

The right adjoint is

$$\Phi_R(E) = Rp_*\big(\mathcal{O}_p(1) \otimes \omega_W[-1] \otimes L\iota^*E\big)\,.$$

The cotwist $\cot_\Phi$ is defined as the cone of the unit morphism

$$\mathrm{id} \to \Phi_R \circ \Phi \to \cot_\Phi\,.$$

A direct calculation shows that $\Phi_R \circ \Phi$ splits as

$$\Phi_R \circ \Phi \cong \mathrm{id} \oplus \omega_C[-2]$$

and $\cot_\Phi$ is isomorphic to $\omega_C[-2]$, which is an auto-equivalence. Thus, Φ is a spherical functor [2, 17, 42] and we obtain an auto-equivalence of $D^b(X)$, the twist t_Φ, defined as the cone of the counit morphism [2, Theorem 1.1]

$$\Phi \circ \Phi_R \to \mathrm{id} \to \mathrm{t}_\Phi\,.$$

We consider an anti-equivalence of order two defined as[5]

$$\rho = \mathrm{t}_\Phi \circ [2] \circ (-)^\vee\,.$$

For any $E \in D^b(X)$ we have the important exact triangle

$$E^\vee[2] \to \rho(E) \to \Phi \circ \Phi_R[1]\big(E^\vee[2]\big)\,. \qquad (\Delta)$$

We can now state the main, and most difficult, result of this section.

Theorem 2.2.

(i) $\rho\big(\mathcal{A}_0\big) \subset \mathcal{A}_0[-1]$,

(ii) $\rho\big(\mathcal{A}_1\big) \subset \mathcal{A}_1$.

[5] The derived dual of Section 1.1 is $\mathbb{D}^X = [2] \circ (-)^\vee$.

Outline. The proof will be given in Sections 2.6 to 2.9. We start with properties and basic results in Sections 2.5, 2.6. In Section 2.7 we prove that objects in $\mathcal{A}$ with support contained in W are successive extensions of objects which are *scheme-theoretically* supported on W. This will also be applied in Section 4 to prove that a function ν defines a stability function on $\mathcal{A}_{\leq 1}$. Theorem 2.2 (i) is proved in Section 2.8. In Section 2.9 we prove that for any $E \in \mathcal{A}_{\leq 1}$ the perverse cohomology sheaves satisfy

$$ {}^p\mathcal{H}^i\big(\rho(E)\big) = 0\,, \quad i \neq 0, 1\,, \quad {}^p\mathcal{H}^1\big(\rho(E)\big) \in \mathcal{A}_0\,. \tag{$*$} $$

This suffices to deduce Theorem 2.2 (ii). □

Theorem 2.2 should remind the reader of an analogous property of the derived dual $\mathbb{D}^X$ acting on $\mathrm{Coh}(X)$:

$$ \mathbb{D}^X(\mathrm{Coh}_0(X)) = \mathrm{Coh}_0(X)[-1]\,, \quad \mathbb{D}^X(\mathrm{Coh}_1(X)) = \mathrm{Coh}_1(X)\,. $$

Indeed, the next section clarifies the origin of this analogy.

2.4 Crepant case

We explain now our main motivation for the tilt $\mathcal{A}$ and for the derived anti-equivalence ρ by considering the case of a type III contraction $X \to Y$, as described in Section 1.2.

In this setting, Y is the coarse moduli space of a Calabi–Yau orbifold $\mathcal{Y}$ that has $B\mathbb{Z}_2$-singularities along a copy of the curve C. The derived categories of X and $\mathcal{Y}$ are isomorphic via the derived McKay correspondence [9]

$$ \Phi\colon D^b(X) \xrightarrow{\sim} D^b(\mathcal{Y})\,. $$

The heart $\mathcal{A} \subset D^b(X)$ coincides with Bridgeland's category of perverse sheaves [7, 51]

$$ \mathcal{A} = {}^0\mathrm{Per}(X/Y)\,, $$

so under the McKay correspondence it should be regarded as $\mathrm{Coh}(\mathcal{Y})$. Indeed, let $j_0\colon C_0 \hookrightarrow Y$ be the contraction of W, i.e. $C_0 = \pi(W)$. Then, for any $T \in \mathrm{Coh}(X)$ the higher pushforward $R^1\pi_*T$ is supported on C_0, so $R^1\pi_*T = 0$ if and only if

$$ 0 = j^*R^1\pi_*T = R^1p_*\iota^*T. $$

The equality used holds by the proper base change theorem.

Under the McKay correspondence, the notion of perverse dimension that we defined coincides with the usual dimension on the orbifold. The anti-equivalence ρ coincides with the derived dual $\mathbb{D}^{\mathcal{Y}} = R\mathcal{H}om(-, \mathcal{O}_{\mathcal{Y}})[2]$ on the orbifold, i.e.:

Proposition 2.3. In the setting above, we have

$$\rho = \Phi^{-1} \circ \mathbb{D}^{\mathcal{Y}} \circ \Phi \,.$$

Proof. We let $\Psi = \Phi^{-1} \circ \mathbb{D}^{\mathcal{Y}} \circ \Phi \circ \rho$. Since Φ is a derived equivalence, whereas ρ and $\mathbb{D}^{\mathcal{Y}}$ are derived anti-equivalences, the composition Ψ is a derived equivalence. We prove that Ψ is isomorphic to the identity by analysing $\Psi(k(x))$ and using again [19, Corollary 5.23].

If $x \in X \setminus W$ then Lemma 2.4 shows that

$$\Psi(k(x)) = \big(\Phi^{-1} \circ \mathbb{D}^{\mathcal{Y}} \circ \Phi\big)\big(k(x)[-1]\big) = \big(\Phi^{-1} \circ \mathbb{D}^{\mathcal{Y}}\big)\big(k(\pi(x)\big)[1] = k(x) \,.$$

For $x \in W$, one has the exact triangle of Lemma 2.4 and applying $\Phi^{-1} \circ \mathbb{D}^{\mathcal{Y}} \circ \Phi$ to it produces the exact triangle

$$\mathcal{O}_B(-1) \to \Psi(k(x)) \to \mathcal{O}_B(-2)[1]. \tag{2.1}$$

We used that $\Phi^{-1} \circ \mathbb{D}^{\mathcal{Y}} \circ \Phi$ is an anti-equivalence and we determine the images of $\mathcal{O}_B(-2), \mathcal{O}_B(-1)[-1]$ using [10, Section 4.3], [5, Appendix A]

$$\Phi\big(\mathcal{O}_B(-2)[1]\big) = \mathcal{O}_p^+ \,, \quad \Phi\big(\mathcal{O}_B(-1)\big) = \mathcal{O}_p^- \,, \quad \mathbb{D}^{\mathcal{Y}}\big(\mathcal{O}_p^\pm\big) = \mathcal{O}_p^\mp[-1] \,.$$

Extensions determined by (2.1) are classified by

$$\mathrm{Hom}(\mathcal{O}_B(-2), \mathcal{O}_B(-1)) \cong \mathbb{C}^2$$

and we get that $\Psi(k(x)) \cong k(f(x))$ for some $f(x) \in B = \pi^{-1}\big(\pi(x)\big)$.

By [19, Corollary 5.23] it follows that $f \colon X \to X$ is an isomorphism and $\Psi = (M \otimes -) \circ f_*$ for some line bundle M. Since $f_{|X \setminus W} = \mathrm{id}_{X \setminus W}$, we conclude that $f = \mathrm{id}$. By Proposition 2.5 and the fact that Φ preserves structure sheaves, one easily sees that $\Psi(\mathcal{O}_X) = \mathcal{O}_X$ and thus M is the trivial line bundle, so $\Psi \cong \mathrm{id}$. □

As we mentioned in Section 1.2, when X is obtained as a crepant resolution our results follow from [5]. The previous proposition explains how the heart $\mathcal{A}$ and the duality ρ play the role of $\mathrm{Coh}(\mathcal{Y})$ and $\mathbb{D}^{\mathcal{Y}}$, respectively, in the proof of the rationality and functional equation for the orbifold PT invariants [5].

2.5 Properties of ρ

We gather here some of the key properties of the duality operator ρ. We begin with a direct computation of the image of some objects (of perverse dimension 0) under ρ.

Lemma 2.4. For all points $x \in X$ and fibers $B \subset W$

(i) If $x \notin W$, then $\rho(k(x)) = k(x)[-1]$,

(ii) $\rho\big(\mathcal{O}_B(-2)[1]\big) = \mathcal{O}_B(-2)$,

(iii) $\rho\big(\mathcal{O}_B(-1)\big) = \mathcal{O}_B(-1)[-1]$,

(iv) if $x \in B$ there is an exact triangle

$$\mathcal{O}_B(-2) \to \rho\big(k(x)\big) \to \mathcal{O}_B(-1)[-1]\,,$$

(v) for all $k \leq -2$, $\rho\big(\mathcal{O}_B(k)[1]\big) \in \mathcal{A}_0[-1]$,

(vi) for all $k \geq -1$, $\rho\big(\mathcal{O}_B(k)\big) \in \mathcal{A}_0[-1]$.

Proof. Part (i) follows from $k(x)^\vee[2] = k(x)[-1]$ and $\Phi_R\big(k(x)\big) = 0$. Part (ii) and (iii) are computed directly. Then, any $x \in B$ corresponds to an exact triangle

$$\mathcal{O}_B(-1) \to k(x) \to \mathcal{O}_B(-2)[1]\,,$$

and application of ρ yields (iv). For (v) and (vi) we can use induction on k to reduce to (ii) and (iii) respectively. □

Proposition 2.5. We have

$$\rho\big(\mathcal{O}_X\big) = \mathcal{O}_X[2]\,, \qquad \rho \circ \rho = \mathrm{id}\ .$$

Proof. The first claim follows from $\Phi_R(\mathcal{O}_X) = 0$, thus

$$\rho(\mathcal{O}_X) = \mathcal{O}_X^\vee[2] = \mathcal{O}_X[2]\,.$$

For the second claim we use the computations in Lemma 2.4 which imply that for all $x \in X$ there is $y \in X$ such that

$$\rho \circ \rho\big(k(x)\big) \cong k(y)\,.$$

Moreover, $x = y$ for $x \in X \setminus W$. Now we apply the general fact [19, Corollary 5.23] that any auto-equivalence Ψ with $\Psi(k(x)) \cong k(f(x))$ is of the form

$$\Psi = (M \otimes -) \circ f_*\,,$$

where $f\colon X \to X$ is an isomorphism and M is a line bundle. Then $f_{|X\setminus W} = \mathrm{id}$, thus $f = \mathrm{id}$, and by the first claim M must be the trivial line bundle. □

The action of ρ on cohomology can be directly computed using the exact triangle (Δ). For our purposes, it suffices to consider objects $E \in \mathcal{A}_{\leq 1}$, in particular $\mathrm{ch}_0(E) = 0$, and $\mathrm{ch}_1(E)$ is a multiple of w. It is convenient to compute the action using $(\mathrm{ch}_1, \mathrm{ch}_2, \chi)$.

Proposition 2.6. The anti-equivalence ρ acts on $(\mathrm{ch}_1, \mathrm{ch}_2, \chi)$ as

$$\big(r\mathsf{w}, \beta, n\big) \overset{\rho}{\longmapsto} \big(r\mathsf{w}, \beta + \big(\mathsf{w}\cdot\beta - (2-2g)r\big)\,\mathsf{b}, -n\big)\,.$$

Proof. First note that for all $V \in D^b(C)$ we have

$$\begin{aligned}\chi\big(\Phi(V)\big) &= \chi\big(\iota_*(\mathcal{O}_p(-1)\otimes p^*V)\big)\\ &= \chi\big(Rp_*(\mathcal{O}_p(-1))\otimes V\big) = 0\,.\end{aligned}$$

Thus, $\chi\big(\rho(E)\big) = \chi\big(E^\vee[2]\big)\big) = -\chi(E)$. Next, we compute the class of $\rho(E)$ for $E = \iota_*\mathcal{O}_p(-1) = \Phi(\mathcal{O}_C)$. We have $r = 1$ and denote by $\beta = \mathrm{ch}_2(E)$. Using $\Phi_R \circ \Phi \cong \mathrm{id} \oplus \omega_C[-2]$ we find (see also the proof of Lemma 2.19)

$$\rho(E) \cong \Phi\big(\omega_C^2 \otimes \det(\mathcal{E}^\vee)\big)\,.$$

Thus, $\mathrm{ch}_1\big(\rho(E)\big) = \mathrm{ch}_1(E)$ and

$$\begin{aligned}\mathrm{ch}_2\big(\rho(E)\big) &= \beta + \big(4g-4+\deg(\mathcal{E}^\vee)\big)\,\mathsf{b}\\ &= \beta + \big(\mathsf{w}\cdot\beta - (2-2g)\big)\,\mathsf{b}\,.\end{aligned}$$

Finally, let $E \in D^b(X)$ with $[E] = (\beta, n) \in N_{\leq 1}(X)$. Then

$$\mathrm{ch}(L\iota^*E) = \mathrm{ch}(E) - \mathrm{ch}(E\otimes\mathcal{O}_X(-W)) = \mathsf{w}\cdot\beta \in N_0(W)\,.$$

Using the triangle (Δ), we find $\mathrm{ch}_1\big(\rho(E)\big) = 0$ and

$$\mathrm{ch}_2\big(\rho(E)\big) = \beta + (\mathsf{w}\cdot\beta)\,\mathsf{b}\,.$$

The three cases together prove the claim by additivity. □

2.6 Basic results (proof of Theorem 2.2)

We start by setting up some notation that will later be useful in the induction process we'll use.

Notation 2.7. Let $\omega \in \mathrm{Amp}(X)$ be an ample class and $E \in \mathrm{Coh}(X)$ with at most 1-dimensional support outside of W. Denote by $\mathrm{ch}_i^\omega(E) = \omega^{3-i}\cdot\mathrm{ch}_i(E)$. We write $\mathrm{ch}^\omega(E') < \mathrm{ch}^\omega(E)$, if

(i) $0 \leq \mathrm{ch}_1^\omega(E') < \mathrm{ch}_1^\omega(E)$, or

(ii) $0 = \mathrm{ch}_1^\omega(E') = \mathrm{ch}_1^\omega(E)$, and $0 \leq \mathrm{ch}_2^\omega(E') < \mathrm{ch}_2^\omega(E)$, or

(iii) $0 = \mathrm{ch}_1^\omega(E') = \mathrm{ch}_1^\omega(E)$, and $0 = \mathrm{ch}_2^\omega(E') = \mathrm{ch}_2^\omega(E)$, and $0 \leq \mathrm{ch}_3^\omega(E') < \mathrm{ch}_3^\omega(E)$.

Then, $\mathrm{ch}^\omega(E) \geq 0$ with equality if and only if $E = 0$. Note that $\mathrm{ch}^\omega(E) > 0$ is minimal if and only if $E \cong k(x)$ for some $x \in X$.

Notation 2.8. For $G', G \in \mathrm{Coh}(W)$ we write $R^1p_*G' < R^1p_*G$ if

(i) $0 \leq \mathrm{rk}\big(R^1p_*G'\big) < \mathrm{rk}\big(R^1p_*G\big)$, or

(ii) $0 = \mathrm{rk}\big(R^1p_*G'\big) = \mathrm{rk}\big(R^1p_*G\big)$, and $\mathrm{len}\big(R^1p_*G'\big) < \mathrm{len}\big(R^1p_*G\big)$, where $\mathrm{len}(-)$ is the length of a 0-dimensional sheaf.

Lemma 2.9. (i) For all $T \in \mathrm{Coh}(X)$

$$R^1p_*L\iota^*T = R^1p_*\iota^*T\,.$$

(ii) There is a short exact sequence

$$0 \to R^1p_*L^{-1}\iota^*(T) \to p_*L\iota^*(T) \to p_*\iota^*(T) \to 0\,.$$

(iii) For all $G \in \mathrm{Coh}(W)$, $L^k\iota^*\iota_*G = 0$ for $k \neq 0, -1$ and

$$L^{-1}\iota^*\iota_*G = \omega_W^\vee \otimes G\,, \quad \iota^*\iota_*G = G\,.$$

Proof. There is a spectral sequence

$$E_2^{k,l} = R^kp_*\mathcal{H}^l(L\iota^*T) \implies R^{k+l}p_*L\iota^*T\,.$$

Since $\dim(p) = 1$, the only non-vanishing term contributing to $R^1p_*L\iota^*T$ is $R^1p_*\iota^*T$ and the differentials vanish. The second statement follows analogously. The third assertion follows from $\iota^*\mathcal{O}_X(-W) = \omega_W^\vee$. ☐

Lemma 2.10.

(i) If $\iota_*G \in \mathcal{F}$, then $p_*G = 0$,

(ii) if $\iota_*G \in \mathcal{A}_{\leq 1}$, then $Rp_*(G) \in \mathrm{Coh}(C)$.

Proof. If $p_*G \neq 0$ we may choose a sufficiently ample $L \in \mathrm{Pic}(C)$ and a non-zero section $L^\vee \to p_*G$. By adjunction we have a non-zero $p^*L^\vee \to G$. This contradicts $\iota_*G \in \mathcal{F}$ because $R^1p_*p^*L^\vee = 0$, i.e. $\iota_*p^*L^\vee \in \mathcal{T}$. The statement (ii) follows from (i) and the definition of $\mathcal{T}$. ☐

Lemma 2.11. (i) For all $G \in \mathrm{Coh}(W)$

$$\mathrm{rk}\big(R^1p_*(\mathcal{O}_p(1) \otimes G)\big) \leq \mathrm{rk}(R^1p_*G)\,,$$

with strict inequality if $\mathrm{rk}(R^1p_*G) > 0$. In that case

$$\mathrm{rk}\big(R^1p_*(\omega_W^\vee \otimes G)\big) < \mathrm{rk}(R^1p_*G)\,.$$

(ii) If $\mathrm{rk}(R^1p_*G) = 0$, then

$$\dim\big(R^1p_*(\mathcal{O}_p(1)\otimes G)\big) \leq \dim(R^1p_*G)\,,$$

with strict inequality if $\dim(R^1p_*G) > 0$. In that case

$$\dim\big(R^1p_*(\omega_W^\vee\otimes G)\big) < \dim(R^1p_*G)\,.$$

Proof. The second assertion follows from the first one since

$$\omega_W = \mathcal{O}_p(-2)\otimes p^*(\omega_C\otimes\det\mathcal{E}^\vee)\,.$$

(i) Denote by $r_k = \mathrm{rk}\big(R^1p_*(\mathcal{O}_p(k)\otimes G)\big)$. Let $C_0\subset W$ be the zero locus of a section of $\mathcal{O}_p(1)$, thus C_0 is a section of the projection p. For all $k\in\mathbb{Z}$ there is a sequence

$$\mathcal{O}_p(k-1)\otimes G\to\mathcal{O}_p(k)\otimes G\to\mathcal{O}_{C_0}(k)\otimes G\to 0\,.$$

Thus, $r_k\leq r_{k-1}$. The Euler sequence on W implies

$$\det(\mathcal{E}^\vee)\otimes R^1p_*(\mathcal{O}_p(k-2)\otimes G)\to\mathcal{E}^\vee\otimes R^1p_*(\mathcal{O}_p(k-1)\otimes G)$$
$$\to R^1p_*(\mathcal{O}_p(k)\otimes G)\to 0\,,$$

thus $r_{k-2}-2\,r_{k-1}+r_k\geq 0$. If $r_{k-1}=r_{k-2}$, then $r_k=r_{k-1}$, thus $r_k=r_0>0$ for all $k\geq 0$. This is a contradiction since $\mathcal{O}_p(1)$ is p-ample and so $r_k=0$ for $k\gg 0$. For (ii) The proof is the same, with rank replaced by the length of 0-dimensional sheaves. □

Recall the sequence from Section 2.1

$$0\to L_1\to\mathcal{E}^\vee\to L_2\to 0\,.$$

Let g be the genus of C and define

$$k_- = -g+\min\{0,\deg(L_1),\deg(L_2)\}-1\,,$$
$$k_+ = -g+\max\{0,\deg(L_1),\deg(L_2)\}+1\,.$$

We have the following technical lemma which we will apply multiple times.

Lemma 2.12. Let $0\neq\iota_*G\in\mathcal{A}_{\leq 1}$. There is a line bundle $L\in\mathrm{Pic}(C)$ and a non-zero morphism $K\to G$ with

$$K=\mathcal{O}_p(-1)\otimes p^*L\,,\ \text{or}\ K=\omega_p\otimes p^*L[1]\,.$$

If $Rp_*G\neq 0$, we may choose L such that

$$k_-+\frac{\chi(G)}{\max\{\mathrm{rk}(Rp_*G),1\}}\leq\chi(L)\leq k_++\frac{\chi(G)}{\max\{\mathrm{rk}(Rp_*G),1\}}\,.$$

If $Rp_*G=0$, we may choose L such that

$$\chi(L)=\chi\big(G\otimes\mathcal{O}_p(1)\big)-1\,.$$

Remark 2.13. Note that if $\iota_* G \in \mathcal{T}_{\leq 1}$, then $K = \mathcal{O}_p(-1) \otimes p^* L$ because

$$\operatorname{Hom}(\mathcal{F}[1], \mathcal{T}) = 0\,.$$

Proof. Recall that $Rp_* G \in \operatorname{Coh}(C)$ is a sheaf by Lemma 2.10, in particular $\operatorname{rk}(Rp_* G) \geq 0$. First, assume that $Rp_* G \neq 0$. Let $M \in \operatorname{Pic}(C)$ with

$$\operatorname{rk}(Rp_* G) \deg(M) < \chi(G)\,,$$

then by Riemann–Roch

$$H^0(C, M^\vee \otimes Rp_* G) \neq 0\,.$$

We may choose M so that $\deg(M)$ is the nearest integer to

$$\frac{\chi(G)}{\max\{\operatorname{rk}(Rp_* G), 1\}} - 1\,.$$

Note that when $\operatorname{rk}(Rp_* G) = 0$ we must have $\chi(G) = \chi(Rp_* G) \geq 0$ since $Rp_* G \in \operatorname{Coh}_0(C)$.

Now we can use the Euler sequence on W which yields an exact triangle in $\mathcal{A}_{\leq 1}$:

$$\mathcal{O}_p(-1) \otimes p^*(\mathcal{E}^\vee \otimes M) \to p^* M \to \omega_p \otimes p^* M[1]\,.$$

Since $\operatorname{Hom}(p^* M, G) = H^0(C, M^\vee \otimes Rp_* G) \neq 0$, we find

$$\operatorname{Hom}(\mathcal{O}_p(-1) \otimes p^*(\mathcal{E}^\vee \otimes M), G) \neq 0\,, \text{ or}$$
$$\operatorname{Hom}(\omega_p \otimes p^* L[1], G) \neq 0\,.$$

In the latter case, set $K = \omega_p \otimes p^* M[1]$ and $L = M$. In the former case we can use the sequence

$$0 \to L_1 \to \mathcal{E}^\vee \to L_2 \to 0$$

and argue as above, i.e. we can set $K = \mathcal{O}_p(-1) \otimes L_i \otimes M$ and $L = L_i \otimes M$ for $i = 1$ or $i = 2$. Since $\chi(M) = \deg(M) + 1 - g$, we find in all three cases the bound stated for $\chi(L)$.

Now assume that $Rp_*(G) = 0$, thus $G \in \mathcal{T}$ is a sheaf. If $G \neq 0$, we may choose a section $j\colon C_0 \hookrightarrow W$ in the linear system $|\mathcal{O}_p(1)|$, such that $j^* G \neq 0$.There is an exact triangle

$$G \to G \otimes \mathcal{O}_p(1) \to j_* L j^*\big(G \otimes \mathcal{O}_p(1)\big)$$

and, since $Rp_*(G) = 0$,

$$p_*\big(G \otimes \mathcal{O}_p(1)\big) \cong p_* j_* L j^*\big(G \otimes \mathcal{O}_p(1)\big)\,,$$

By Lemma 2.9, the latter surjects onto $p_* j_* j^*\big(G \otimes \mathcal{O}_p(1)\big)$ which is non-zero since C_0 is a section of p. Now apply the first part to $G \otimes \mathcal{O}_p(1)$ to obtain a non-zero $p^* L \to G \otimes \mathcal{O}_p(1)$ and twist by $\mathcal{O}_p(-1)$. $\square$

2.7 Support (proof of Theorem 2.2)

Lemma 2.14. For all $T \in \mathcal{T}$ there are $T', T'' \in \mathcal{T}$ and an exact sequence

$$0 \to T' \to T \to T'' \to 0\,,$$

such that

(i) $T' \in \mathrm{Coh}_{\leq 1}(X)$ and $\iota^* T' \in \mathrm{Coh}_0(W)$,

(ii) $\mathrm{supp}(T'')_{red} \subset W$.

Proof. Let $\mathrm{supp}(T)_{red} = Z \cup W'$ with $W' \subset W$, $\dim(Z) \leq 1$ and $Z \cap W$ empty or 0-dimensional. By a standard argument, we can find such a sequence with $\mathrm{supp}_{red}(T') \subset Z$ and $\mathrm{supp}_{red}(T'') \subset W$, see e.g. [43, Tag 01YD]. Then, $T' \in \mathcal{T}$ is immediate from the definition and $T'' \in \mathcal{T}$ since $\mathcal{T}$ is closed under quotients. □

The rest of this section concerns sheaves with set-theoretic support contained in W. Given full subcategories $\mathcal{C}_i \subset D^b(X)$ we denote by $\langle \{\mathcal{C}_i\}_i \rangle_{\mathrm{ex}}$ the smallest extension-closed subcategory of $D^b(X)$ containing each $\mathcal{C}_i$.

Proposition 2.15. Let $T \in \mathcal{T}$, $F \in \mathcal{F}$, and $B \subset W$ be a fiber of the projection p. Then,

(i) If $\mathrm{supp}(T)_{red} \subset W$, then $T \in \langle \mathcal{T} \cap \iota_* \mathrm{Coh}(W) \rangle_{\mathrm{ex}}$,

(ii) if $\mathrm{supp}(T)_{red} \subset B$, then $T \in \langle \mathcal{T} \cap \iota_* \mathrm{Coh}(B) \rangle_{\mathrm{ex}}$,

(iii) $F \in \langle \mathcal{F} \cap \iota_* \mathrm{Coh}(W) \rangle_{\mathrm{ex}}$,

(iv) if $\mathrm{supp}(F)_{red} \subset B$, then $F \in \langle \mathcal{F} \cap \iota_* \mathrm{Coh}(B) \rangle_{\mathrm{ex}}$.

Proof of Proposition 2.15 (i), (ii). Let $T \in \mathcal{T}$ with $\mathrm{supp}(T)_{red} \subset W$, then there is an exact sequence

$$0 \to T' \to T \to \iota_* \iota^* T \to 0\,,$$

with T' a quotient of $T \otimes \mathcal{O}_X(-W)$. Note that $\iota_* \iota^* T \in \mathcal{T}$ as it is a quotient of T. It follows from Lemma 2.11 that $T \otimes \mathcal{O}_X(-W) \in \mathcal{T}$, thus $T' \in \mathcal{T}$. The sequence implies $\mathrm{ch}^\omega(T') < \mathrm{ch}^\omega(T)$, see Notation 2.7. Since $\mathrm{ch}^\omega(T) = 0$ if and only if $T = 0$, we conclude by induction.

The analogous argument proves (ii). By (i) we may consider sheaves $\iota_* G \in \mathcal{T}$ with $\mathrm{supp}(G)_{red} \subset B$. Let $j \colon B \hookrightarrow W$, then we have an exact sequence

$$0 \to G' \to G \to j_* j^* G \to 0\,,$$

with G' a quotient of $G \otimes \mathcal{O}_W(-B)$. Since $\iota_*\big(G \otimes \mathcal{O}_W(-B)\big) \in \mathcal{T}$ we can conclude as above. □

For the proofs of (iii) and (iv) we require the following results. Recall the Notation 2.8.

Lemma 2.16. For all $F \in \mathcal{F}$ there are $F', F'' \in \mathcal{F}$ and an exact sequence

$$0 \to F' \to F \to F'' \to 0$$

such that

(i) $F'' \cong \iota_*\iota^*F''$,

(ii) $R^1p_*\iota^*F \xrightarrow{\sim} R^1p_*\iota^*F''$,

(iii) $R^1p_*(\iota^*F') < R^1p_*(\iota^*F)$.

Proof. Consider the restriction $F \twoheadrightarrow \iota_*\iota^*F$ and the decomposition

$$0 \to T \to \iota_*\iota^*F \to F'' \to 0$$

obtained from the torsion pair $(\mathcal{T}, \mathcal{F})$. Since $\mathcal{F}$ is closed under subobjects, we obtain the desired sequence of sheaves in $\mathcal{F}$. Property (i) follows since F'' is a quotient of $\iota_*\iota^*F$. For (ii) note that $\iota^*F = \iota^*\iota_*\iota^*F$ and, as consequence of the definition of the torsion pair $(\mathcal{T}, \mathcal{F})$, the map $\iota_*\iota^*F \twoheadrightarrow F''$ induces an isomorphism on $R^1p_*\iota^*$.

For (iii) we consider the pullback $L\iota^*$ of the sequence and apply Rp_* to obtain

$$p_*L\iota^*F'' \to R^1p_*\iota^*F' \to R^1p_*\iota^*F \to R^1p_*\iota^*F'' \to 0\,.$$

The last map is an isomorphism, thus the first one must be surjective. By Lemma 4.3, $p_*\iota^*F'' = 0$ and by Lemma 2.9:

$$p_*L\iota^*F'' = R^1p_*(L\iota^{-1}F'') = R^1p_*(\omega_W^\vee \otimes \iota^*F'')\,.$$

Lemma 2.11 together with (ii) implies (iii). □

Lemma 2.17. For all $\iota_*G \in \mathcal{F}$ supported on finitely many fibers of p, there exists a fiber $j\colon B_y \hookrightarrow W$ and $\iota_*G', \iota_*G'' \in \mathcal{F}$ and an exact sequence

$$0 \to G' \to G \to G'' \to 0$$

such that

(i) $G'' \cong j_*j^*G''$,

(ii) $R^1p_*G \otimes k(y) \xrightarrow{\sim} R^1p_*G'' \otimes k(y)$,

(iii) $R^1p_*G' < R^1p_*G$.

Proof. The proof is parallel to the proof of Lemma 2.16. □

Proof of Proposition 2.15 (iii), (iv). To prove (iii) we use Lemma 2.16 and induction to reduce to $F \in \mathcal{F}$ with $R^1p_*(\iota^*F) = 0$. But then $F \in \mathcal{F} \cap \mathcal{T}$, thus $F = 0$. The analogous argument proves (iv). □

2.8 Zero-dimensional perverse sheaves (proof of Theorem 2.2)

We use a generating set of objects with extension closure $\mathcal{A}_0$ to prove Theorem 2.2 (i).

Lemma 2.18. Denote the fibers of the projection by $B_y = p^{-1}(y)$, then

(i) $\mathcal{A}_0 \cap \mathcal{F}[1] = \big\langle \{\mathcal{O}_{B_y}(k)[1] : y \in C\,, k \leq -2\}\big\rangle_{\mathrm{ex}}$,

(ii) $\mathcal{A}_0 \cap \mathcal{T} = \big\langle \mathrm{Coh}_0(X)\,, \{\mathcal{O}_{B_y}(k) : y \in C\,, k \geq -1\}\big\rangle_{\mathrm{ex}}$.

Proof. By Proposition 2.15 (iv), $\mathcal{A}_0 \cap \mathcal{F}[1]$ is the extension closure of shifted sheaves $G[1]$ supported on a single fiber $j\colon B_y \hookrightarrow W$. Then $p_* j_* G = 0$ by Lemma 4.3, thus decomposing G into a 0-dimensional sheaf and a sum of line bundles we find that G is torsion-free and only line bundles $\mathcal{O}_{B_y}(k)$ with $k \leq -2$ appear.

For (ii) use Lemma 2.14 and Proposition 2.15 (ii) to reduce to $\mathrm{Coh}_0(X)$ and sheaves supported on some $j\colon B_y \hookrightarrow W$. Decomposing the latter into a sum of a 0-dimensional sheaf and line bundles $\mathcal{O}_{B_y}(k)$, we must have $k \geq -1$. □

Theorem 2.2 (i) now follows from Lemmas 2.4 and 2.18.

2.9 One-dimensional perverse sheaves (proof of Theorem 2.2)

Let $F \in \mathcal{F}$. By Lemma 2.16 we may assume that $F \cong \iota_* G$ is supported on W. The proof of Lemma 4.3 showed that we have an injection

$$G \hookrightarrow \omega_p \otimes p^* V\,,$$

where $V = R^1 p_* G$. Let T be the cokernel. The inclusion induces an isomorphism on $R^1 p_*(-)$, thus $\iota_* T \in \mathcal{T}_{\leq 1}$. We have an exact triangle

$$T \to G[1] \to \omega_p \otimes p^* V[1]\,.$$

To prove Theorem 2.2 (ii) it suffices to consider sheaves in $\mathcal{T}_{\leq 1}$ and objects of the form $\iota_*\big(\omega_p \otimes p^* V\big)[1]$.

Recall the functor $\Phi\colon D^b(C) \to D^b(X)$ defined as

$$\Phi(V) = \iota_*\big(\mathcal{O}_p(-1) \otimes p^* V\big)\,.$$

For any $E \in D^b(X)$ we have an exact triangle (Δ)

$$E^\vee[2] \to \rho(E) \to \Phi \circ \Phi_R[1]\big(E^\vee[2]\big)\,.$$

We consider the long exact sequence of cohomology sheaves for the *standard t-structure* associated to this triangle. Let $\mathcal{H}^i = \mathcal{H}^i\big(\rho(E)\big)$. Property $(*)$ is equivalent to

$$\mathcal{H}^{-1} \in \mathcal{F}\,, \quad \mathcal{H}^1 \in \mathcal{T} \cap \mathcal{A}_0\,, \quad \mathcal{H}^i = 0\,, \quad i \neq -1, 0, 1\,.$$

Lemma 2.19. Let $V \in \mathrm{Coh}(C)$, then

(i) $\rho\Big(\iota_*\big(\omega_p \otimes p^*V\big)[1]\Big)$ satisfies Property $(*)$,

(ii) $\rho\Big(\iota_*\big(\mathcal{O}_p(-1) \otimes p^*V\big)\Big)$ satisfies Property $(*)$.

Proof. Denote by $E = \iota_*\big(\omega_p \otimes p^*V\big)[1]$, then

$$E^\vee[2] = \iota_*\big(p^*(\omega_C \otimes V^\vee)\big)\,.$$

Note that $V^\vee = R\mathcal{H}om(V, \mathcal{O}_C)$ has cohomology sheaves

$$\mathcal{H}^0\big(V^\vee\big) \in \mathrm{Coh}_1(C)\,, \quad \mathcal{H}^1\big(V^\vee\big) \in \mathrm{Coh}_0(C)\,, \quad \mathcal{H}^i\big(V^\vee\big) = 0\,, \quad i \neq 0, 1\,.$$

Then, we find that

$$\mathcal{H}^0\big(E^\vee[2]\big) \in \mathcal{T}_{\leq 1}\,, \quad \mathcal{H}^1\big(E^\vee[2]\big) \in \mathcal{T}_{\leq 1} \cap \mathcal{A}_0\,, \quad \mathcal{H}^i\big(E^\vee[2]\big) = 0\,, \quad i \neq 0, 1\,.$$

Direct computation yields

$$\Phi_R[1]\big(E^\vee[2]\big) = p_*\big(\mathcal{O}_p(1)\big) \otimes \omega_C \otimes V^\vee\,,$$

with cohomology sheaves

$$\begin{aligned}&\mathcal{H}^0\big(\Phi \circ \Phi_R[1](E^\vee[2])\big) \in \mathcal{T}_{\leq 1}\,, \quad \mathcal{H}^1\big(\Phi \circ \Phi_R[1](E^\vee[2])\big) \in \mathcal{T}_{\leq 1} \cap \mathcal{A}_0\,,\\ &\mathcal{H}^i\big(\Phi \circ \Phi_R[1](E^\vee[2])\big) = 0\,, \quad i \neq 0, 1\,.\end{aligned}$$

Together this proves (i). For (ii) let $E = \Phi(V)$, then

$$E^\vee[2] = \iota_*\big(\mathcal{O}_p(-1) \otimes p^*(\omega_C \otimes \det(\mathcal{E}^\vee) \otimes V^\vee)\big)[1] = \Phi\big(\widetilde{V}[1]\big)\,,$$

where $\widetilde{V} = \omega_C \otimes \det(\mathcal{E}^\vee) \otimes V^\vee$. Using $\Phi_R \circ \Phi \cong \mathrm{id} \oplus \omega_C[-2]$ we obtain a split exact triangle

$$\rho(E) \to \Phi\big(\widetilde{V}[2] + \omega_C \otimes \widetilde{V}\big) \to \Phi(\widetilde{V})[2]\,.$$

Thus, $\rho\big(E\big) \cong \Phi\big(\omega_C \otimes \widetilde{V}\big)$, which satisfies

$$\begin{aligned}&\mathcal{H}^0\big(\Phi(\omega_C \otimes \widetilde{V})\big) \in \mathcal{T}_{\leq 1}\,, \quad \mathcal{H}^1\big(\Phi(\omega_C \otimes \widetilde{V})\big) \in \mathcal{T}_{\leq 1} \cap \mathcal{A}_0\,,\\ &\mathcal{H}^i\big(\Phi(\omega_C \otimes \widetilde{V})\big) = 0\,, \quad i \neq 0, 1\,.\end{aligned}$$

□

Proposition 2.20. For all $E \in \mathrm{Coh}_{\leq 1}(X) \cap \mathcal{T}$ the image $\rho(E)$ satisfies Property $(*)$.

Proof. We decompose E with respect to the torsion pair $(\mathcal{A}_0, \mathcal{A}_1)$ of $\mathcal{A}_{\leq 1}$. The $\mathcal{A}_0$ part is covered by Theorem 2.2 (i). Thus, assume

$$E \in \mathrm{Coh}_{\leq 1}(X) \cap \mathcal{T} \cap \mathcal{A}_1 \,,$$

in particular $E \in \mathrm{Coh}_1(X)$. We apply Lemma 2.14 to E. First assume that $\iota^* E \in \mathrm{Coh}_0(W)$. It follows from purity of E that $L\iota^* E = \iota^* E$. Dualizing, we have $E^\vee[2] \in \mathrm{Coh}_1(X) \cap \mathcal{A}_1$ and

$$L\iota^*\big(E^\vee[2]\big) = \iota^*\big(E^\vee[2]\big) \in \mathrm{Coh}_0(W) \,.$$

We have the exact triangle (Δ)

$$E^\vee[2] \to \rho(E) \to \Phi \circ \Phi_R[1]\big(E^\vee[2]\big) \,.$$

The left and right objects are sheaves in $\mathcal{T}_{\leq 1}$, thus $\rho(E) \in \mathcal{T}_{\leq 1}$ as well.

It remains to prove Property $(*)$ for sheaves $E = \iota_* G \in \mathrm{Coh}_1(X)$. Let $\mathcal{H}^i = \mathcal{H}^i\big(\rho(E)\big)$, we must prove that

$$\mathcal{H}^{-1} \in \mathcal{F} \,, \quad \mathcal{H}^1 \in \mathcal{T} \cap \mathcal{A}_0 \,, \quad \mathcal{H}^i = 0 \,, \quad i \neq -1, 0, 1 \,.$$

Note that $Rp_*(L\iota^* E^\vee[2]) = p_*(L\iota^* E^\vee[2])$ lies in $D^{[-1,0]}(C)$, thus

$$\mathcal{H}^i\big(\Phi \circ \Phi_R[1](E^\vee[2])\big) = 0 \,, \qquad i \neq -1, 0 \,.$$

Thus, in fact $\mathcal{H}^i = 0$ for $i \neq -1, 0$ from the long exact sequence. We must argue that $\mathcal{H}^{-1} \in \mathcal{F}$. Note that $E^\vee[2] = \iota_*(G^\vee \otimes \omega_W[1])$. We have an exact sequence of sheaves

$$0 \to \mathcal{H}^{-1} \to \mathcal{O}_p(-1) \otimes p^* p_*\big(G^\vee \otimes \omega_W[1] \otimes \mathcal{O}_p(1)\big) \to \iota_*(G^\vee \otimes \omega_W[1]) \,.$$

Let $L \in \mathrm{Pic}(C)$ be a line bundle. Since $Rp_* \mathcal{O}_W = \mathcal{O}_C$ we have

$$\begin{aligned} &\mathrm{Hom}\Big(\mathcal{O}_p(-1) \otimes p^* L, \mathcal{O}_p(-1) \otimes p^* p_*\big(G^\vee \otimes \omega_W[1] \otimes \mathcal{O}_p(1)\big)\Big) \\ &\cong \mathrm{Hom}\Big(L, p_*\big(G^\vee \otimes \omega_W[1] \otimes \mathcal{O}_p(1)\big)\Big) \\ &\cong \mathrm{Hom}\Big(\mathcal{O}_p(-1) \otimes p^* L, \iota_*(G^\vee \otimes \omega_W[1])\Big) \,. \end{aligned}$$

Thus,

$$\mathrm{Hom}\Big(\mathcal{O}_p(-1) \otimes p^* L, \mathcal{H}^{-1}\Big) = 0 \,,$$

and $\mathcal{H}^{-1} \in \mathcal{F}$ by Lemma 2.12. □

Lemma 2.21.

$$\mathcal{T}_{\leq 1} = \Big\langle \mathrm{Coh}_{\leq 1}(X) \cap \mathcal{T}\,, \Phi\big(\mathrm{Coh}_1(C)\big) \Big\rangle_{\mathrm{ex}} .$$

Proof. The inclusion "$\supset$" is immediate. By Lemma 2.14 and Proposition 2.15 we know that $\mathcal{T}_{\leq 1}$ is the extension closure of sheaves $T \in \mathrm{Coh}_{\leq 1}(X)$ with $\iota^* T \in \mathrm{Coh}_0(W)$ and pushforwards $\iota_*(G) \in \mathcal{T}_{\leq 1}$. Thus, it suffices to consider sheaves $T = \iota_* G$. Let

$$T_0 \to T \to T_1$$

be the decomposition in $\mathcal{A}$ with respect to the torsion pair $(\mathcal{A}_0\,, \mathcal{A}_1)$. Since $T \in \mathrm{Coh}(X)$ we have $\mathrm{Hom}(\mathcal{F}[1], T) = 0$, thus

$$T_0 \in \mathrm{Coh}(X) \cap \mathcal{A}_0 = \mathcal{T}_0 .$$

This category is closed under quotients. Thus, replacing T_0 by its image, we may assume that $T_0 \to T$ is an injection of sheaves. It follows that $T_1 \in \mathrm{Coh}(X) \cap \mathcal{A}_1 = \mathcal{T}_1$. We have $\mathcal{T}_0 \subset \mathrm{Coh}_{\leq 1} \cap \mathcal{T}$, thus we may assume $\iota_* G \in \mathcal{T}_1$.

By Lemma 2.12 there is a line bundle L and a non-zero morphism

$$\mathcal{O}_p(-1) \otimes p^* L \to G .$$

Taking the image and cokernel of this map, we obtain an exact sequence of sheaves in $\mathcal{T}_{\leq 1}$

$$0 \to \iota_* G' \to \iota_* G \to \iota_* G'' \to 0 ,$$

such that $0 \neq \iota_*(G') \in \mathcal{A}_1$. If

$$\mathcal{O}_p(-1) \otimes p^* L \twoheadrightarrow G'$$

is an isomorphism, then $\iota_* G' \in \Phi\big(\mathrm{Coh}_1(C)\big)$. Otherwise, G' has dimension at most one. By Proposition 4.3 (i) we have[6] $\ell(\iota_* G') > 0$, thus

$$\ell(\iota_* G) > \ell(\iota_* G'') \geq 0 .$$

By Proposition 4.3 (iii) we have $\ell(\iota_* G'') = 0$ if and only if $\iota_* G'' \in \mathcal{A}_0$, so we can conclude by induction. $\square$

Proposition 2.22. For all $T \in \mathcal{T}_{\leq 1}$ the image $\rho(T)$ satisfies Property $(*)$.

Proof. Follows from Lemma 2.19, Proposition 2.20, and Lemma 2.21. $\square$

[6] Here we use $\ell(-)$ as defined in Section 4.2. The properties proved in Proposition 4.3 do not depend on Lemma 2.21.

Proof of Theorem 2.2 (ii). The results of this section imply that for all $E \in \mathcal{A}$ the image $\rho(E)$ satisfies Property $(*)$. Let $E \in \mathcal{A}_1$ and $Q \in \mathcal{A}_0$. Then, $\rho(Q) \in \mathcal{A}_0[-1]$ by Theorem 2.2 (i) and, by purity of E,

$$\mathrm{Hom}(\rho(E), Q[-1]) = \mathrm{Hom}(\rho(Q)[1], E) = 0\,.$$

Thus, $\rho(E) \in \mathcal{A}_{\leq 1}$. But then $\rho(E) \in \mathcal{A}_1$ because

$$\mathrm{Hom}(Q, \rho(E)) = \mathrm{Hom}(E, \rho(Q)) = 0\,,$$

since $\mathrm{Hom}^k(E, F) = 0$ for all $E, F \in \mathcal{A}$ and $k < 0$. $\square$

3 Hall algebras, pairs, and wall-crossing

3.1 Numerical Grothendieck groups

The numerical Grothendieck group $N(X)$ is the Grothendieck group of $D^b(X)$ modulo the Euler paring. We will tacitly use the injection into the even cohomology via the Chern character. The class $[E] \in N(X)$ is equivalently characterised by

$$\big(\mathrm{ch}_0(E), \mathrm{ch}_1(E), \mathrm{ch}_2(E), \chi(E)\big)\,.$$

The numerical Grothendieck group admits a dimension filtration $N_{\leq k}(X)$. For our purposes, we define $N_{\leq k}$ as the numerical Grothendieck group of $\mathcal{A}_{\leq k}$. We will only consider objects of perverse dimension ≤ 1. Explicitly,

$$N_0 = \mathbb{Z} \cdot \mathsf{b} \oplus \mathbb{Z} \cdot \mathsf{p}\,, \quad N_{\leq 1} = \mathbb{Z} \cdot \mathsf{w} \oplus N_{\leq 1}(X)\,,$$

where b and w are the classes of a fiber resp. the divisor as introduced in Section 1.1 and $N_0(X) \cong \mathbb{Z}$ is spanned by the point class p. We also define $N_1 = N_{\leq 1}/N_0$ and we choose a splitting

$$N_{\leq 1} = N_0 \oplus N_1\,.$$

An element $\alpha \in N_{\leq 1}$ can be written as

$$\alpha = (\gamma, c) = (r\mathsf{w}, \beta + j\mathsf{b}, n)$$

where $\gamma = (r, \beta) \in N_1$ and $c = (j, n) \in N_0$.

We will consider various generating series of DT invariants using the Novikov parameter z of $\mathbb{Q}[[N_{\leq 1}]]$ and we use the notation

$$Q = z^{\mathsf{b}}\,, \quad -q = z^{\mathsf{p}}\,, \quad t = z^{[\mathcal{O}_X]}\,.$$

In particular, for α as above $z^\alpha = z^\gamma\,(-q)^n\,Q^j$.

3.2 Hall algebra

We briefly recall the notion of Hall algebras following [50]. Let $\mathcal{C} \subset D^b(X)$ be the heart of a bounded t-structure. In our applications we use two different hearts:

$$\mathcal{C} = \left\langle \mathrm{Coh}_{\geq 2}[1], \mathrm{Coh}_{\leq 1} \right\rangle \text{ and } \mathcal{C} = \left\langle \mathcal{A}_{\geq 2}[1], \mathcal{A}_{\leq 1} \right\rangle.$$

The first is used to define PT and BS invariants, the second is used to define ${}^p\mathrm{PT}$ invariants. Both of these hearts are open by [5, Lemma 4.1] so they satisfy the technical hypothesis in [5, Appendix B], [6, Section 3].

The objects of $\mathcal{C}$ form an algebraic stack which we still denote by $\mathcal{C}$ and we assume that it is an open substack of the stack $\mathcal{M}$ of objects

$$\left\{ E \in D^b(X) : \mathrm{Ext}^{<0}(E,E) = 0 \right\}.$$

The Hall algebra $\mathrm{H}(\mathcal{C})$ is the $\mathbb{Q}$-vector space generated by maps of algebraic stacks $[\mathcal{Z} \to \mathcal{C}]$, where $\mathcal{Z}$ is an algebraic stack of finite type with affine stabilizers, modulo some motivic relations described in [50].

The Hall algebra $\mathrm{H}(\mathcal{C})$ admits a product induced by extensions and, via cartesian products, is a module over $K(\mathrm{St}/\mathbb{C})$, the Grothendieck ring of stacks with affine stabilizers. Equivalently,

$$K(\mathrm{St}/\mathbb{C}) = K(\mathrm{Var}/\mathbb{C})[\mathbb{L}^{-1}, (\mathbb{L}^n - 1)^{-1}]$$

where $\mathbb{L} = [\mathbb{A}^1 \to \operatorname{Spec} \mathbb{C}]$. The decomposition

$$\mathcal{C} = \coprod_{\alpha \in N(X)} \mathcal{C}_\alpha$$

into numerical classes induces a decomposition of the Hall algebra

$$\mathrm{H}(\mathcal{C}) = \bigoplus_\alpha \mathrm{H}_\alpha(\mathcal{C}).$$

The feature of most interest in the Hall algebra is the existence of the integration map. To state this we introduce two more definitions. We let $\mathrm{H}^{\mathrm{reg}}(\mathcal{C}) \subset \mathrm{H}(\mathcal{C})$ be the $K(\mathrm{Var}/\mathbb{C})[\mathbb{L}^{-1}]$-submodule spanned by $[Z \to \mathcal{C}]$ so that Z is a variety and

$$\mathrm{H}^{\mathrm{sc}}(\mathcal{C}) = H^{\mathrm{reg}}(\mathcal{C})/(\mathbb{L}-1)\mathrm{H}^{\mathrm{reg}}(\mathcal{C}).$$

This has the structure of a Poisson algebra. The integration map maps $\mathrm{H}^{\mathrm{sc}}(\mathcal{C})$ to the Poisson torus

$$\mathbb{Q}[N(X)] = \bigoplus_{\alpha \in N(X)} \mathbb{Q} z^\alpha.$$

The Poisson torus has the structure of a Poisson algebra as well. Its bracket is defined by

$$\{z^\alpha, z^{\alpha'}\} = (-1)^{\chi(\alpha,\alpha')} \chi(\alpha, \alpha') z^{\alpha+\alpha'}.$$

Theorem 3.1 ([50, Theorem 2.8]). There is a Poisson algebra homomorphism

$$I\colon \mathrm{H}^{\mathrm{sc}}(\mathcal{C}) \to \mathbb{Q}[N(X)]$$

such that if Z is a variety and $f\colon Z \to \mathcal{C}_\alpha \hookrightarrow \mathcal{C}$ then

$$I([Z \xrightarrow{f} \mathcal{C}]) = \left(\int_Z f^*\nu_{\mathcal{C}}\right) z^\alpha$$

where $\nu_{\mathcal{C}}$ is the Behrend function on the stack $\mathcal{C}$.

The Hall algebra can be enlarged to the graded pre-algebra $H^{\mathrm{gr}}(\mathcal{C})$ by defining its generators to be $[\mathcal{Z} \to \mathcal{X}]$ with $\mathcal{Z}$ being an algebraic stack with affine stabilizers such that $\mathcal{Z}_\alpha$ is of finite type for every $\alpha \in N(X)$ (instead of asking that $\mathcal{Z}$ is already of finite type). One can define analogous versions $H^{\mathrm{gr,reg}}(\mathcal{C}), H^{\mathrm{gr,sc}}(\mathcal{C})$. The integration map extends to

$$I : H^{\mathrm{gr,sc}}(\mathcal{C}) \to \mathbb{Q}\{N(X)\}.$$

3.3 Pairs

We consider various notions of stable objects in $D^b(X)$ and their associated generating series. All of them are defined via a pair of subcategories $(\mathcal{T}, \mathcal{F})$ of either $\mathrm{Coh}_{\leq 1}$ or $\mathcal{A}_{\leq 1}$. We consider the categories

$$\mathcal{B} = \left\langle \mathcal{O}_X[1]\,, \mathrm{Coh}_{\leq 1} \right\rangle, \quad {}^p\mathcal{B} = \left\langle \mathcal{O}_X[1]\,, \mathcal{A}_{\leq 1} \right\rangle.$$

Definition 3.2 ([5, Definition 3.9]). An object $P \in \mathcal{B}$ or $P \in {}^p\mathcal{B}$ is called a $(\mathcal{T}, \mathcal{F})$-pair if

(i) $\mathrm{rk}(P) = -1$,

(ii) $\mathrm{Hom}(T, P) = 0$ for all $T \in \mathcal{T}$,

(iii) $\mathrm{Hom}(P, F) = 0$ for all $F \in \mathcal{F}$.

The class of P is $(-1, \alpha)$ with $\alpha \in N_{\leq 1}$. The notion of $(\mathcal{T}, \mathcal{F})$-pairs for fixed α defines a stack $\mathrm{Pairs}(\mathcal{T}, \mathcal{F})_\alpha$ which is of finite type in all of our applications and defines an element in the Hall algebra (Lemmas 4.15, 4.19 and 5.1).

In Section 5 we consider BS and PT pairs which are defined in $\mathcal{B}$. Sections 6 and 7 concern pairs defined in ${}^p\mathcal{B}$. The categories $(\mathcal{T}, \mathcal{F})$ arise in two ways:

(i) As torsion pairs associated to a stability function, or

(ii) in the passage of one torsion pair to another, i.e. when crossing a wall.

In the former case, the stability function is ν in Section 6 and ζ in Section 7. In the latter case, given two torsion pairs $(\mathcal{T}_\pm, \mathcal{F}_\pm)$ on different sides of a wall (and sufficiently close to the wall), we consider $(\mathcal{T}_+, \mathcal{F}_-)$. Joyce's wall-crossing formula yields the comparison between pairs on either side of the wall via semistable objects on $\mathcal{W} = \mathcal{T}_- \cap \mathcal{F}_+$.

3.4 Joyce's wall-crossing formula

Let $(\mathcal{T}_\pm, \mathcal{F}_\pm)$ be two torsion pairs and $\mathcal{W} = \mathcal{T}_- \cap \mathcal{F}_+$ be as above. When all the terms are defined, we have an identity in the Hall algebra

$$[\mathcal{W}] * [\mathrm{Pairs}(\mathcal{T}_-, \mathcal{F}_-)] = [\mathrm{Pairs}(\mathcal{T}_+, \mathcal{F}_+)] * [\mathcal{W}]\,.$$

The "no-poles" theorem by Joyce [22, Theorem 8.7] and Behrend-Ronagh [6, Theorems 4, 5] tells us that in adequate conditions

$$(\mathbb{L} - 1)\log(\mathcal{W}) \in H^{\mathrm{gr,sc}}(\mathcal{C})$$

and, therefore,

$$w = I\big((\mathbb{L} - 1)\log(\mathcal{W})\big) \in \mathbb{Q}\{N(X)\}$$

is well-defined. The conditions that guarantee this are the following:

(i) $\mathcal{W}_\alpha$ is an algebraic stack of finite type,

(ii) $\mathcal{W}$ is closed under extensions and direct summands,

(iii) for every $\alpha \in N(X)$ there are finitely many ways to decompose $\alpha = \alpha_1 + \ldots + \alpha_n$ such that $\mathcal{W}_{\alpha_i} \neq \emptyset$.

When all these conditions are satisfied (including the moduli of pairs defining elements in the Hall algebra), we say that the pairs $(\mathcal{T}_\pm, \mathcal{F}_\pm)$ are wall-crossing material. When this happens, we have Joyce's wall-crossing formula which we will repeatedly use:

$$I\big((\mathbb{L} - 1)\mathrm{Pairs}(\mathcal{T}_+, \mathcal{F}_+)\big) = \exp\big(\{w, -\}\big) \circ I\big((\mathbb{L} - 1)\mathrm{Pairs}(\mathcal{T}_-, \mathcal{F}_-)\big)\,.$$

3.5 Rational functions

In this paper we repeatedly encounter series expansions of rational functions

$$f \in \mathbb{Q}(N_0) = \mathbb{Q}(q, Q)\,.$$

The "direction" of the expansion will play an important role, especially in the ζ-wall-crossing in Section 7. We make here precise what "direction" means.

Given a non-zero linear function $L\colon N_0 \to \mathbb{R}$, we say that a set $S \subset N_0$ is L-bounded if for every $M \in \mathbb{R}$, the set

$$\#\{c \in S\colon L(c) < M\}$$

is finite. Given L, we can define a completion $\mathbb{Q}[N_0]_L$ of $\mathbb{Q}[N_0]$ to be the set of formal power series

$$\sum_{c \in N_0} a_c z^c$$

such that $\{c\colon a_c \neq 0\}$ is L-bounded. The product of power series is well-defined in this completion. Given a rational function $f = g/h$ with $g, h \in \mathbb{Q}[N_0] = \mathbb{Q}[q, Q]$, we say that $F \in \mathbb{Q}[N_0]_L$ is the expansion of f with respect to L if $hF = g$ in the ring $\mathbb{Q}[N_0]_L$.

We briefly go over the different choices of L used throughout the paper and clarify the statements of our results. The series PT_β for usual stable pairs invariants or BS_β for Bryan–Steinberg invariants (see Section 5) can be defined in the completion $\mathbb{Q}[N_0]_L$ where

$$L(j, n) = j + \varepsilon\, n$$

for $0 < \varepsilon \ll 1$. The generating series of perverse stable pairs ${}^p\mathrm{PT}_\gamma$ is defined in the completion $\mathbb{Q}[N_0]_d$ where

$$d(j, n) = 2n + j.$$

In particular, the precise formulation of Theorem 1.2 is that ${}^p\mathrm{PT}_\gamma$ is the expansion of the rational function f_γ with respect to d. Theorem 1.3 is to be understood in $\mathbb{Q}(q, Q)$: the left and right hand side are the expansions of the same rational function in different directions.

This re-expansion in different directions is fundamental in Section 7. There, we will introduce series ${}^p\mathrm{DT}_\gamma^{\zeta,(\mu,\infty)}$ that interpolate between each side of Theorem 1.3: they will be the expansion of the same rational function f_γ with respect to

$$L_\mu(j, n) = 2n + j + \frac{j}{\mu a_0}.$$

Note that L_μ for $\mu \gg 1$ is equivalent to d and for $\mu \ll 1$ it is equivalent to the linear function used for PT or BS.

4 Stability

We use three different stability functions to define stable pairs and study their wall-crossing:

(i) For Bryan–Steinberg stable pairs in Section 5 we use

$$\mu^A \colon \mathrm{Coh}_{\leq 1}(X) \setminus \{0\} \to (-\infty, +\infty] \times (-\infty, +\infty]\,.$$

(ii) For perverse stable pairs in Section 6 we use

$$\nu \colon \mathcal{A}_{\leq 1} \setminus \{0\} \to (-\infty, +\infty]\,.$$

(iii) For the BS/^{p}PT wall-crossing in Section 7 we use

$$\zeta \colon \mathcal{A}_{\leq 1} \setminus \{0\} \to (-\infty, +\infty] \times (-\infty, +\infty]\,.$$

We comment on (i) in Section 4.1. The necessary results about μ^A-stability were proved by Bryan–Steinberg [11] and require only minor modification for our setting. For (ii) we give full proofs in Sections 4.2, 4.3, and 4.5. We also observe in Section 4.4 that $\mathcal{A}_{\leq 1}$ and ν-stability can be obtained from a weak stability condition through a tilting process. Finally, for (iii) we can employ the techniques used for (ii) in a similar way to study ζ-stability. We briefly comment on this in Section 4.7.

4.1 Bryan–Steinberg stability

Let Y be the coarse moduli space of an orbifold Calabi–Yau 3-fold satisfying the *hard Lefschetz condition* and let

$$\pi \colon X \to Y$$

be the distinguished crepant resolution [9, 14]. Denote by $\widetilde{H} \in \mathrm{Nef}(X)$ the pullback of an ample class on Y, and let $\omega \in \mathrm{Amp}(X)$ be ample such that $\omega - \widetilde{H}$ is ample as well. Bryan–Steinberg [11] introduce a function on $\mathrm{Coh}_{\leq 1}(X)$ defined as

$$\mu^{\pi}(E) = \left(\frac{\chi(E)}{\widetilde{H} \cdot \mathrm{ch}_2(E)}, \frac{\chi(E)}{\omega \cdot \mathrm{ch}_2(E)} \right).$$

They are able to prove the necessary technical results [11, Theorem 38, Lemma 47, Lemma 51] which allow to employ Joyce's Hall algebra machinery. We can use the exact same pathway. Critically, we do not require a projection $X \to Y$, the existence of a nef class $A \in \mathrm{Nef}(X)$ as described in the condition ($\diamondsuit$) suffices. We then define μ^A by the same formula as μ^{π}, replacing $\widetilde{H}$ by A. The proofs in [11] carry over to our setting where a projection π does not necessarily exist:

Proposition 4.1 ([11])**.** The slope μ^A defines a stability condition on $\mathrm{Coh}_{\leq 1}(X)$. Moreover, the moduli stack of μ^A-semistable sheaves $\mathcal{M}^{\mu^A}_{(\beta,n)}$ is a finite type open substack of the moduli stack $\mathcal{M}$ parametrizing perfect complexes $E \in D^b(X)$ with $\mathrm{Ext}^{<0}(E,E) = 0$.

Proof. As we pointed out already, the proofs of Theorem 38 and Lemma 47 carry over verbatim to show that μ^A is a stability condition and that the family of sheaves in $\mathcal{M}^{\mu^A}_{(\beta,n)}$ is bounded. The fact that $\mathcal{M}^{\mu^A}_{(\beta,n)}$ is a finite type open substack of $\mathcal{M}$ then follows from [45, Theorem 3.20]. □

4.2 Nironi stability

Recall the nef class $A \in \mathrm{Nef(X)}$ and $a_0 \in \mathbb{Z}_{>0}$ such that $\iota^* A$ is numerically equivalent to $a_0\mathsf{b}$. Let g be the genus of the curve C. For $E \in \mathcal{A}_{\leq 1}$ with

$$\big(\mathrm{ch}_1(E), \mathrm{ch}_2(E), \chi(E)\big) = \big(r\mathsf{w}, \beta, n\big)$$

define the slope $\nu\colon \mathcal{A}_{\leq 1} \setminus \{0\} \to \mathbb{Q} \cup \{+\infty\}$ as $\nu(E) = \dfrac{d(E)}{\ell(E)}$, where

$$\begin{aligned} d(E) &= r(1-g) + 2\,n - \frac{1}{2}\,\mathsf{w}\cdot\beta\,, \\ \ell(E) &= 2\,A\cdot\beta + r\,a_0\,. \end{aligned}$$

Note that for $G \in \mathrm{Coh}(W)$, by Grothendieck–Riemann–Roch

$$A \cdot \mathrm{ch}_2(\iota_* G) = a_0\,\mathrm{rk}(Rp_* G)\,.$$

In the crepant case, the class A can be taken as the pullback of an ample class from the coarse moduli space Y and the stability matches the notion of Nironi's slope stability [36] on $\mathrm{Coh}_{\leq 1}(\mathcal{Y})$.

Recall that Nironi's slope stability is defined in the analogous way, using a self-dual generating bundle V and the modified Hilbert polynomial

$$p_E(k) = \chi\big(V, E \otimes \mathcal{O}_X(A)^k\big) = \ell(E)\,k + d(E)\,.$$

Our definition resembles this notion replacing V by the ρ-invariant K-theory class of $\mathcal{O}_X \oplus \mathcal{O}_X(W/2)$ and replacing the Euler pairing by the Mukai pairing.

Example 4.2. To illustrate ν for zero-dimensional perverse sheaves, consider a skyscraper sheaf $k(x)$ and the perverse sheaves $\mathcal{O}_B(-2)[1]$ and $\mathcal{O}_B(-1)$ supported on a fiber $B = p^{-1}(y)$. In the crepant case, these objects correspond to a non-stacky point, and the stacky points $\mathcal{O}_y^+$ and $\mathcal{O}_y^-$ respectively [10, Section 4.3]. In all three cases $\ell(-) = 0$ and the computation for $d(-)$ is

$$\begin{aligned} d(k(x)) &= 0 + 2 - 0 = 2\,, \\ d(\mathcal{O}_B(-2)[1]) &= -d(\mathcal{O}_B(-2)) = -(0 - 2 + 1) = 1\,, \\ d((\mathcal{O}_B(-1)) &= 0 + 0 + 1 = 1\,. \end{aligned}$$

Proposition 4.3.

(i) For all $T \in \mathcal{T}_{\leq 1}$ set-theoretically supported on W we have $\ell(T) \geq 0$, with equality if and only if $T \in \mathcal{T}_0$.

(ii) For all $F \in \mathcal{F}_{\leq 1}$ we have $\ell(F) \leq 0$, with equality if and only if $F \in \mathcal{F}_0$.

(iii) For all $E \in \mathcal{A}_{\leq 1}$ we have $\ell(E) \geq 0$, with equality if and only if $E \in \mathcal{A}_0$. In that case, $d(E) \geq 0$, with equality if and only if $E = 0$.

Proof. For (i) and (ii) we may apply Proposition 2.15 and assume that T and F are scheme-theoretically supported on W, i.e. we consider pushforwards $\iota_* G$ with $G \in \mathrm{Coh}(W)$.

(i) Let $\iota_* G \in \mathcal{T}_{\leq 1}$. Since $R^1 p_* G = 0$ we have $\mathrm{rk}(Rp_* G) \geq 0$, thus both summands of $\ell(\iota_* G)$ are non-negative, and $\ell(\iota_* G) = 0$ if and only if $r = 0$ and $A \cdot \mathrm{ch}_2(\iota_* G) = 0$, thus $\iota_* G \in \mathcal{T}_0$.

(ii) Let $\iota_* G \in \mathcal{F}$. We claim that $r \leq \mathrm{rk}(R^1 p_* G)$. Let $V = R^1 p_* G$ and consider the map $Rp_* G \to V[-1]$ which lifts to $G \to \omega_p \otimes p^* V$. Let Ker and Im be the kernel and image, i.e.

$$\mathrm{Ker} \to G \to \mathrm{Im}\,.$$

Since $\mathcal{F}$ is closed under subobjects, $\mathrm{Ker} \in \mathcal{F}$. Since $\mathrm{Im} \subset \omega_p \otimes p^* V$ we have $p_* \mathrm{Im} = 0$. The isomorphism

$$R^1 p_* G \cong V$$

factors through $R^1 p_* \mathrm{Im}$. We find that $R^1 p_* \mathrm{Ker} = 0$, thus

$$\iota_* \mathrm{Ker} \in \mathcal{F} \cap \mathcal{T} = 0$$

and $G \hookrightarrow \omega_p \otimes p^* V$ must be injecive, which implies $r \leq \mathrm{rk}(R^1 p_* G)$ by comparing ranks. Since $p_* G = 0$ by Lemma 4.3, this implies

$$\ell(\iota_* G) = -2a_0 \,\mathrm{rk}(R^1 p_* G) + r\, a_0 \leq -a_0 \,\mathrm{rk}(R^1 p_* F) \leq 0\,.$$

From this we get $\ell(\iota_* G) = 0$ if and only if $\mathrm{rk}(R^1 p_* G) = 0$ and then $r = 0$, thus $\iota_* G \in \mathcal{F}_0$.

(iii) By Lemma 2.14, Proposition 2.15, and (i)-(ii) it remains to consider $E \in \mathrm{Coh}_{\leq 1}(X)$ such that $\iota^* E \in \mathrm{Coh}_0(W)$. Then, by condition ($\diamondsuit$)

$$\ell(E) = 2A \cdot \mathrm{ch}_2(E) \geq 0\,,$$

with equality if and only if $\mathrm{ch}_2(E) \in \mathbb{Z}_{\geq 0} \cdot \mathsf{b}$. Since $\mathsf{w} \cdot \mathrm{ch}_2(E) \geq 0$, whereas $\mathsf{w} \cdot \mathsf{b} = -2$, we must in fact have $\mathrm{ch}_2(E) = 0$, i.e. E is a 0-dimensional sheaf.

For the positivity of $d(-)$ on $\mathcal{A}_0$ we may use Lemma 2.18. If $E \in \mathrm{Coh}_0(X)$ then $d(E) = 2\chi(E) \geq 0$. Moreover, we can compute directly

$$d(\mathcal{O}_B(k)) = 2k + 3 > 0 \text{ for } k \geq -1$$
$$d(\mathcal{O}_B(k)[1]) = -(2k + 3) > 0 \text{ for } k \leq -2.$$ □

Proposition 4.4. The slope ν defines a stability condition on $\mathcal{A}_{\leq 1}$:

(i) ν satisfies the see-saw property,

(ii) Harder–Narasimhan filtrations exist.

Proof. The proof of the see-saw property is standard, so it is enough to prove that $\mathcal{A}_{\leq 1}$ is ν-Artinian.

Suppose that $E_1 \supseteq E_2 \supseteq \ldots$ in $\mathcal{A}_{\leq 1}$. Then $\ell(E_i)$ is a decreasing sequence bounded below by 0, so it must stabilize. Thus, for large enough i the cone $C(E_{i+1} \to E_i) \in \mathcal{A}$ must be in $\mathcal{A}_0$ so $\nu(E_{i+1}) \leq \nu(E_i)$. □

Proposition 4.5. The slope ν satisfies

$$\nu\big(\rho(E)\big) = -\nu(E)\,, \quad \nu\big(E \otimes \mathcal{O}_X(A)\big) = \nu(E) + 1\,.$$

Proof. The equality $\ell\big(\rho(E)\big) = \ell(E)$ is clear since $A \cdot \mathsf{b} = 0$. Using Proposition 2.6 and $\mathsf{w} \cdot \mathsf{b} = -2$ we have

$$\begin{aligned} d\big(\rho(E)\big) &= r(1-g) - 2n - \frac{1}{2}\mathsf{w} \cdot \big(\beta + (\mathsf{w} \cdot \beta - (2-2g)r)\,\mathsf{b}\big) \\ &= -r(1-g) - 2n + \frac{1}{2}\mathsf{w} \cdot \beta = -d(E)\,. \end{aligned}$$

For the second equality, a computation using $A^2 \cdot \mathsf{w} = 0$ shows that

$$\ell\big(E \otimes \mathcal{O}_X(A)\big) = \ell(E)\,, \quad d\big(E \otimes \mathcal{O}_X(A)\big) = d(E) + \ell(E)\,.$$ □

Definition 4.6. An object $E \in \mathcal{A}_{\leq 1}$ is called ν-stable (resp. semistable), if for all non-trivial subobjects $F \to E$ in $\mathcal{A}_{\leq 1}$ we have $\nu(F) < \nu(E)$ (resp. $\nu(F) \leq \nu(E)$).

The following lemma will be useful in Section 4.6.

Lemma 4.7. Let $L \in \operatorname{Pic}(C)$, then

(i) $\iota_*\big(\mathcal{O}_p(-1) \otimes p^*L\big)$ is ν-stable of slope $\chi(L) + \frac{1}{2}\deg(\mathcal{E}) + 1 - g$,

(ii) $\iota_*\big(\omega_p \otimes p^*L[1]\big)$ is ν-stable of slope $\chi(L)$.

Proof. Let E be the object in (i) or (ii). Note that $\ell(E) = 1$ in both cases. From the description of $\mathcal{A}_0$ in Lemma 2.4 we see that E is torsion-free in $\mathcal{A}_{\leq 1}$, i.e. $\operatorname{Hom}(\mathcal{A}_0, E) = 0$. Let

$$E' \to E \to E''$$

be an exact triangle in $\mathcal{A}_{\leq 1}$. Then $\ell(E') = 1$ and $\ell(E'') = 0$, therefore $d(E'') \geq 0$ with equality if and only if $E'' = 0$. Thus, either $\nu(E') < \nu(E)$ or $E' = E$. The slope $\nu(E)$ is easily computed. □

4.3 Curve classes

We denote by N_1^{eff} the image of $\mathcal{A}_{\leq 1}$ in N_1. By Lemma 2.14 and Proposition 2.15, N_1^{eff} is the cone generated by classes $[E]$ where E is from one of the three sets

$$\begin{aligned} S_1 &= \big\{E \in \operatorname{Coh}_{\leq 1}(X) \colon \iota^*E \in \operatorname{Coh}_0(W)\big\}\,, \\ S_2 &= \mathcal{T} \cap \iota_*\operatorname{Coh}(W)\,, \\ S_3 &= \big(\mathcal{F} \cap \iota_*\operatorname{Coh}(W)\big)[1]\,. \end{aligned}$$

Let $\Delta \subset N_1^{\mathrm{eff}}$ be the cone generated by classes form S_2 and S_3.

Lemma 4.8. For any $l > 0$, the set

$$\{\gamma \in \Delta : \ell(\gamma) \leq l\}$$

is finite.

Proof. It suffices to prove the claim for classes $[E]$ with E from either set S_2 or S_3. Consider $\iota_*G \in S_2$. Recall that $\ell(\iota_*G) = a_0\,(2\operatorname{rk}(Rp_*G) + r)$ and, because $\iota_*G \in \mathcal{T}$, we have $\operatorname{rk}(Rp_*G) = \operatorname{rk}(p_*G) \geq 0$. So there are only finitely many possibilities for r and for $A \cdot \operatorname{ch}_2(\iota_*G)$. Since $N_1(W)$ has rank 2, the map

$$N_1(W)_{\mathbb{Q}} \,/\, \mathbb{Q} \cdot \mathsf{b} \xrightarrow{A\cdot} \mathbb{Q}$$

is an isomorphism, showing that there are finitely many possibilities for $\operatorname{ch}_2(\iota_*G)$ in $N_1(X)/\mathbb{Z} \cdot \mathsf{b}$.

The argument for S_3 is similar to S_2. Indeed, for $\iota_*G[1] \in S_3$ Lemma 4.3 (ii) implies that

$$-A \cdot \mathrm{ch}_2(\iota_*(G)) = -a_0 \mathrm{rk}(Rp_*G) = a_0 \mathrm{rk}(R^1 p_* G)$$

is bounded (recall that $p_*G = 0$ by Lemma 2.10 (i)), so again there are finitely many possibilities for both $A \cdot \mathrm{ch}_2(\iota_* G)$ and r. □

We say that a decomposition $\gamma = \sum \gamma_i$ is effective if all $\gamma_i \in N_1^{\mathrm{eff}}$.

Corollary 4.9. There are only finitely many effective decompositions of $\gamma \in N_1^{\mathrm{eff}}$.

Proof. Every effective decomposition of γ is a sum

$$\gamma = \gamma' + \gamma''$$

with $\gamma' \in N_1^{\mathrm{eff}}$ a sum of classes from S_1, and $\gamma'' \in \Delta$. In particular, γ' is an effective curve class and $\ell(\gamma'') \leq \ell(\gamma)$. By Lemma 4.8 there are finitely many such classes γ''. By standard arguments [28, Corollary 1.19], there are finitely many decompositions of γ' into effective curve classes. □

4.4 Weak stability

In this section we connect ν-stability to the notion of weak stability in the sense of Toda [46]. We obtain an alternative description of the category $\mathcal{A}_{\leq 1}$. This section does not contain any results which are strictly necessary for the remainder of the paper and it rather serves as a comparison. In [41, 45] the authors study the moduli problem for (weak) stability conditions on tilted hearts. They are able to prove that the two key properties, *generic flatness*, and *boundedness of semistable objects* are preserved, in some sense, under a tilting process

$$(Z, \mathcal{C}) \rightsquigarrow (Z^\dagger, \mathcal{C}^\dagger)\,.$$

It seems likely that this technique can be employed to deduce the results in Section 4.6, although we will not pursue it in this paper.

Let $\mathcal{C}_{\leq 1} = \mathrm{Coh}_{\leq 1}(X/W)$ be the category of coherent sheaves which are at most 1-dimensional outside of W. This category was studied in [49] for Calabi–Yau 3-folds containing an embedded $\mathbb{P}^2$. The numerical K-group of $\mathcal{C}_{\leq 1}$ is the same as that of $\mathcal{A}_{\leq 1}$

$$N_0 = \mathbb{Z} \cdot \mathsf{b} \oplus \mathbb{Z} \cdot \mathsf{p}\,, \quad N_{\leq 1} = \mathbb{Z} \cdot \mathsf{w} \oplus N_{\leq 1}(X)\,.$$

We can define a weak stability function $Z = (Z_0, Z_1)$ associated to the filtration

$$0 \subset N_0 \subset N_{\leq 1}\,.$$

Let $\omega \in \operatorname{Amp}(X)$ be an ample class. For $E \in \mathcal{C}_{\leq 1}$ define

$$Z_1(E) = -\ell(E) + i\,\omega^2 \cdot \operatorname{ch}_1(E)\,,$$
$$Z_0(E) = -d(E) + i\,\omega \cdot \operatorname{ch}_2(E)\,.$$

Here, $d(E)$ and $\ell(E)$ are as defined in Section 4.2. If $[E] \in N_0$, set $Z(E) = Z_0(E)$, otherwise $Z(E) = Z_1(E)$. Then, for all $0 \neq E \in \mathcal{C}_{\leq 1}$:

(i) $Z(E) \in \mathbb{H} \cup \mathbb{R}_{<0}$,

(ii) E admits a Harder–Narasimhan filtration.

Property (i) follows from condition ($\diamondsuit$). Property (ii) holds because $\mathcal{C}_{\leq 1}$ is Noetherian and the image of Z is discrete.[7]

Now we consider a tilting process

$$(Z, \mathcal{C}_{\leq 1}) \rightsquigarrow (Z^\dagger, \mathcal{C}^\dagger_{\leq 1})\,.$$

Define the generalized slope of $0 \neq E \in \mathcal{C}_{\leq 1}$ as

$$\lambda(E) = -\frac{\operatorname{Re} Z(E)}{\operatorname{Im} Z(E)} \in (-\infty, \infty]\,.$$

This leads to the standard construction of a torsion pair

$$\mathcal{T}_\lambda = \big\langle \lambda\text{-semistable } E \in \mathcal{C}_{\leq 1} \text{ with } \lambda(E) \geq 0 \big\rangle\,,$$
$$\mathcal{F}_\lambda = \big\langle \lambda\text{-semistable } E \in \mathcal{C}_{\leq 1} \text{ with } \lambda(E) < 0 \big\rangle\,.$$

Define the tilt as

$$\mathcal{C}^\dagger_{\leq 1} = \big\langle \mathcal{F}_\lambda[1], \mathcal{T}_\lambda \big\rangle\,,$$

and the function

$$Z^\dagger(E) = -d(E) + i\,\ell(E)\,.$$

Proposition 4.3 has two consequences. Firstly, the pair $(\mathcal{T}_\lambda\,, \mathcal{F}_\lambda)$ agrees with the perverse torsion pair:

$$\mathcal{T}_\lambda = \mathcal{T}_{\leq 1}\,, \quad \mathcal{F}_\lambda = \mathcal{F}\,.$$

In particular, $\mathcal{A}_{\leq 1} = \mathcal{C}^\dagger_{\leq 1}$. Secondly, we have for all $0 \neq E \in \mathcal{A}_{\leq 1}$

$$Z^\dagger(E) \in \mathbb{H} \cup \mathbb{R}_{<0}\,.$$

Harder–Narasimhan filtrations exist by Proposition 4.4. The associated slope function of $Z^\dagger$ is precisely ν. In particular, $Z^\dagger$-semistability coincides with ν-semistability. Note that this resembles the standard way to interpret slope stability on curves as Bridgeland stability [7, Example 5.4], [32]. We have obtained $\mathcal{A}_{\leq 1}$ and ν-stability through a tilting process from $(Z, \mathcal{C}_{\leq 1})$. We do not know if this fits the general framework of tilting process established in [41].

[7] We have not checked the support property for Z. It might be possible to give a proof following the arguments in the surface case [4, Section 4].

4.5 Boundedness

In this section we prove some boundedness and finiteness results that will be needed to ensure that the moduli stacks of ν-semistable sheaves are finite type, see Proposition 4.15. This property is necessary for the application and analysis of the wall-crossing formula and for the proof of rationality in Section 6. For $E \in \mathcal{A}_{\leq 1}$ we denote by $\nu_+(E)$, $\nu_-(E)$ the maximal and minimal slopes of the Harder–Narasimhan factors with respect to ν-stability. For $I \subset \mathbb{R} \cup \{+\infty\}$ denote by $\mathcal{M}^\nu(I)$ the stack of all $E \in \mathcal{A}_{\leq 1}$ such that all HN-factors have slope contained in I. If $I = [\delta_-, \delta_+]$, this is equivalent to $\nu_+(E) \leq \delta_+$ and $\nu_-(E) \geq \delta_-$. The substack $\mathcal{M}^\nu_\gamma(I)$ parametrizes all such E with fixed $[E] = \gamma \in N_1$. The special case $I = [\delta, \delta]$ parametrizes ν-semistable E of slope δ and is denoted $\mathcal{M}^\nu_\gamma(\delta)$. The substack $\mathcal{M}^\nu_{(\gamma,c)} \subset \mathcal{M}^\nu_\gamma(\delta)$ corresponds to a fixed class $(\gamma, c) \in N_{\leq 1}$. We write c_E to denote the class of $[E]$ in N_0.

Proposition 4.10. Let $I \subset \mathbb{R}$ be a bounded interval and $E \in \mathcal{M}^\nu_\gamma(I)$. There exists a finite subset $S \subset N_0$ depending on γ and I such that $c_E \in S$, if one of the following holds:

(i) $E \in \mathrm{Coh}_{\leq 1}(X)$ with $\iota^* E \in \mathrm{Coh}_0(W)$,

(ii) $E \cong \iota_* \iota^* E$.

Proof. (i) In the first case, $\mathrm{ch}_2(E) \in N_1^{\mathrm{eff}}(X)$ is an effective curve class with residue $\gamma \in N_1^{\mathrm{eff}}$. The class $\gamma + j\mathsf{b}$ is effective only for finitely many negative values of j. On the other hand, note that for any $E \in \mathrm{Coh}_{\leq 1}(X)$ with $\iota^* E \in \mathrm{Coh}_0(W)$ we have $\mathrm{ch}_2(E) \cdot \mathsf{w} \geq 0$. If $j \gg 0$, then $\mathsf{w} \cdot (\gamma + j\mathsf{b}) < 0$, since $\mathsf{w} \cdot \mathsf{b} = -2$. Thus, j must lie in a bounded interval, so we have finitely many curve classes $\mathrm{ch}_2(E)$. Recall that by definition of $\nu(E)$ we have

$$\chi(E) = \frac{1}{2}\big(\ell(E) \cdot \nu(E) + \frac{1}{2}\mathsf{w} \cdot \mathrm{ch}_2(E)\big)\,.$$

Since $\nu(E) \in I$, also $\chi(E)$ lies in a bounded interval.

(ii) Let $I \subset [\delta_-, \delta_+]$ and $G = \iota^* E$. We first prove that $\chi(E)$ is bounded below. For this we may assume $\chi(E) < 0$. By Lemma 2.9 we have $Rp_*(G) \in \mathrm{Coh}(C)$ and also

$$\chi(E) = \chi(G) = \chi(Rp_* G)\,.$$

Let $L \in \mathrm{Pic}(C)$ with

$$\mathrm{rk}(Rp_* G)(\chi(L) + 1 - g) > \chi(G)\,.$$

We may choose $\chi(L) = \chi(G) + g$. Then by Riemann–Roch

$$0 \neq H^1(Rp_* G \otimes L^\vee \otimes \omega_C) = \mathrm{Hom}(Rp_* G, L)\,.$$

The latter is isomorphic to $\mathrm{Hom}(G, \omega_p \otimes p^*L[1])$ by adjunction. The object $\iota_*\big(\omega_p \otimes p^*L[1]\big)$ is stable by Lemma 4.7, with slope $\chi(L)$. Since $\nu_-(E) \geq \delta_-$, we must have $\chi(G) \geq \delta_- - g$.

Now we prove that $\chi(E)$ is bounded above. For this we may assume $\chi(E) > 0$, in particular $Rp_*G \neq 0$. By Lemma 2.12 we obtain $L \in \mathrm{Pic}(C)$ with

$$k_- + \frac{\chi(G)}{\max\{\mathrm{rk}(Rp_*G), 1\}} \leq \chi(L) \leq k_+ + \frac{\chi(G)}{\max\{\mathrm{rk}(Rp_*G), 1\}}$$

and a non-zero morphism $K \to G$ with

$$K = \mathcal{O}_p(-1) \otimes p^*L\,, \text{ or } K = \omega_p \otimes p^*L[1]\,.$$

The object $\iota_*K \in \mathcal{A}_{\leq 1}$ is stable by Lemma 4.7, with slope

$$\nu(\iota_*K) = \chi(L) + \frac{1}{2}\deg(\mathcal{E}) + 1 - g\,, \text{ or } \nu(\iota_*K) = \chi(L)\,.$$

Since $E \in \mathcal{M}^{\nu}_{\gamma}(I)$ it follows that $\nu(K) \leq \delta_+$. But if $\chi(G) \gg 0$ we get $\chi(L) \gg 0$ (recall that $a_0\mathrm{rk}(Rp_*G) = A \cdot \mathrm{ch}_2(E)$ only depends on γ) and thus $\nu(K) \gg 0$, a contradiction.

We conclude that $\chi(G)$ is bounded. By the same argument as in (i), since $\nu(E) = d(E)/\ell(E) \in I$ is also bounded we can show that there are only finitely many possibilities for j in $\mathrm{ch}_2(E) = \beta + j\mathsf{b}$, finishing the proof. □

We can now prove the boundedness of certain families of objects in $\mathcal{A}_{\leq 1}$. The underlying notion of sheaf of t-structures is established in [1] which we apply to the heart of perverse t-structure $\mathcal{A} \subset D^b(X)$. For a discussion of bounded families see [45, Section 3]. We will repeatedly use the following useful result [45, Lemma 3.16] which relies on the finite dimensionality of Ext^1-groups.

Lemma 4.11. Let $\mathcal{S}_i$ be sets of objects in $D^b(X)$ for $i = 1, 2, 3$ such that S_1, S_2 are bounded. Assume that for any object $E_3 \in \mathcal{S}_3$ there are $E_i \in \mathcal{S}_i$ for $i = 1, 2$ and an exact triangle

$$E_1 \to E_3 \to E_2\,.$$

Then, $\mathcal{S}_3$ is also bounded.

First, we consider the family of zero-dimensional perverse sheaves.

Lemma 4.12. Let $D \geq 0$ and $\mathcal{S}$ be the family of $E \in \mathcal{A}_0$ with $d(E) \leq D$. Then, $\mathcal{S}$ is a bounded family.

Proof. By Lemma 2.18, every $E \in \mathcal{A}_0$ admits a quotient $E \to Q$ in $\mathcal{A}_0$ where Q is one of the following objects:

$$k(x)\,, \quad \mathcal{O}_{B_y}(k-1)\,, \quad \mathcal{O}_{B_y}(-k-2)[1]\,.$$

Here, $x \in X$ is a point, $B_y = p^{-1}(y)$ a fiber of p, and $k \geq 0$. By Lemma 4.3 we have $0 < d(Q) \leq d(E)$, in particular $0 \leq k \leq d(E)$. The family of such objects Q is bounded. We can conclude by induction and Lemma 4.11. □

We can now prove the following result.

Proposition 4.13. Let $I \subset \mathbb{R}$ be a bounded interval and $\gamma \in N_1$. Let $\mathcal{S}$ be one of the following families of objects in $\mathcal{M}^{\nu}_{\gamma}(I)$:

(i) the set of $E \in \mathrm{Coh}_{\leq 1}(X)$ with $\iota^* E \in \mathrm{Coh}_0(W)$,

(ii) the set of $E \cong \iota_* \iota^* E$.

Then, $\mathcal{S}$ is a bounded family.

Proof. (i) Let $I \subset [\delta_-, \delta_+]$ and let $\omega \in \mathrm{Amp}(X)$ be an ample class. We consider ω-slope stability on $\mathrm{Coh}_{\leq 1}(X)$ defined by

$$\mu_\omega(E) = \frac{\chi(E)}{\omega \cdot \mathrm{ch}_2(E)}\,.$$

By Proposition 4.10 (i), the set of curve classes $\mathrm{ch}_2(E)$ for $E \in \mathcal{S}$ is finite, so we can define

$$m_- = \min_{E \in \mathcal{S}} \left\{ \frac{1}{2} \mathsf{w} \cdot \mathrm{ch}_2(E) \right\}, \quad m_+ = \max_{E \in \mathcal{S}} \left\{ \frac{1}{2} \mathsf{w} \cdot \mathrm{ch}_2(E) \right\}.$$

Let $F \subset E$ be a subsheaf and $E \twoheadrightarrow Q$ a quotient, then

$$\chi(F) \leq \frac{1}{2}\big(\ell(F) \cdot \delta_+ + m_+\big)\,,$$
$$\chi(Q) \geq \frac{1}{2}\big(\ell(Q) \cdot \delta_- + m_-\big)\,,$$

Recall that $\mathrm{Coh}_0(X) \subset \mathcal{T}_0$, thus E is torsion-free and $\omega \cdot \mathrm{ch}_2(F) > 0$. By Lemma 4.3 we have $0 \leq \ell(F), \ell(Q) \leq \ell(E)$ and so we obtain a bounded interval J (depending only on γ and I) such that for all E as above, the HN-factors of E with respect to μ_ω-stabilty have slope contained in J. Boundedness of the family of such E now follows from boundedness of μ_ω-stability [20, Theorem 3.3.7].

(ii) Assume that $E \cong \iota_* \iota^* E$ and denote by $G = \iota^* E$. By Proposition 4.10 (ii) the set of classes $\alpha = [E] \in N_{\leq 1}$ for $E \in \mathcal{S}$ is finite. Fix one such α. We use

Lemma 2.12 to obtain $L \in \mathrm{Pic}(C)$ with $\chi(L) \geq n(\alpha)$ bounded below by some $n(\alpha) \in \mathbb{Z}$ determined from the class $\alpha \in N_{\leq 1}$. We have a non-zero morphism

$$K \to G$$

such that K is either $\mathcal{O}_p(-1) \otimes p^*L$ or $\omega_p \otimes p^*L[1]$. In both cases, K is stable by Lemma 4.7. Let G' be the image of this morphism in $\mathcal{A}_{\leq 1}$, thus we obtain an exact triangle with pushforward in $\mathcal{A}_{\leq 1}$

$$G' \to G \to G''.$$

Note that $\mathrm{Hom}(\mathcal{A}_0, \iota_* G) = 0$ since $\iota_* G \in \mathcal{M}^{\nu}_{\alpha}(I)$ and $I \subset \mathbb{R}$ is finite. Thus, $\ell(\iota_* G') > 0$ by Lemma 4.3. We can now bound the slopes of the HN-factors of G' and G'' as follows. There are obvious inequalities

$$\nu_+(\iota_* G') \leq \nu_+(\iota_* G), \quad \nu_-(\iota_* G'') \geq \nu_-(\iota_* G).$$

Since G' is a quotient of K, we get $\nu_-(\iota_* G') \geq \nu(K)$, which is bounded below via $\chi(L) \geq n(\alpha)$ and Lemma 4.7. Thus, $d(\iota_* G') = \ell(\iota_* G')\nu(\iota_* G')$ lies in a bounded interval determined by α and then the same is true for $\iota_* G''$. We can conclude by induction on $\ell(\iota_* G)$ and Lemma 4.11. The case $\ell(\iota_* G) = 0$ is covered by Lemma 4.12. □

4.6 Moduli stacks

The goal of this section is to explain the existence of finite type moduli spaces of ν-semistable objects and stable pairs. The setup is as follows.

Let $\mathcal{A} = \langle \mathcal{F}[1], \mathcal{T} \rangle$ be the category of perverse sheaves defined in Section 2 as the tilt along the torsion pair $(\mathcal{T}, \mathcal{F})$ of $\mathrm{Coh}(X)$. We consider another torsion pair $(\mathcal{T}_{\leq 1}, \mathcal{F}')$ of $\mathrm{Coh}(X)$, where $\mathcal{T}_{\leq 1} = \mathcal{T} \cap \mathcal{A}_{\leq 1}$ and $\mathcal{F}' = \mathcal{T}_{\leq 1}^{\perp}$. Define the tilt

$$\mathrm{Coh}^{\dagger}(X) = \left\langle \mathcal{F}'[1], \mathcal{T}_{\leq 1} \right\rangle.$$

Recall Lieblich's [29] moduli stack $\mathcal{M}$ of objects $E \in D^b(X)$ with

$$\mathrm{Ext}^{<0}(E, E) = 0.$$

The stack $\mathcal{M}$ is an Artin stack locally of finite type.

Lemma 4.14. The stacks of objects $\mathrm{Obj}(\mathrm{Coh}^{\dagger}(X))$ and $\mathrm{Obj}(\mathcal{A})$ define open substacks of $\mathcal{M}$.

Proof. In both cases, the heart is defined as a tilt along a torsion pair. The torsion part is defined by the condition $R^1 p_* L\iota^* = 0$, see Lemma 2.9. This an open condition in families. The torsion-free part of the torsion pair is defined as the orthogonal complement, which is an open condition as well. Then, also the tilt defines an open substack [3, Theorem A.8]. □

We consider stable pairs in the subcategory

$$ {}^p\mathcal{B} = \Big\langle \mathcal{O}_X[1]\,, \mathcal{A}_{\leq 1} \Big\rangle \subset \mathrm{Coh}^\dagger(X)\,. $$

It follows from the argument in [46, Lemma 3.5, Lemma 3.8] that ${}^p\mathcal{B}$ is a Noetherian abelian category. Note that $\mathrm{Coh}^\dagger(X)$, however, is not Noetherian. Let $\mathrm{Obj}^{\geq -1}({}^p\mathcal{B})$ be the substack of objects of rank ≥ -1, thus the rank is either -1 or 0.

Proposition 4.15. Let $I \subset \mathbb{R}$ be an interval, $\delta \in \mathbb{R}$, $\gamma \in N_1$, and $\alpha \in N_{\leq 1}$, then

(i) $\mathrm{Obj}^{\geq -1}({}^p\mathcal{B}) \subset \mathcal{M}$ is an open substack,

(ii) $\mathcal{M}^\nu_\gamma(I) \subset \mathrm{Obj}(\mathcal{A}_{\leq 1})$ is an open substack. If I is bounded, $\mathcal{M}^\nu_\gamma(I)$ is an Artin stack of finite type,

(iii) $\mathcal{M}^\nu_\alpha([\delta, +\infty])$ and $\mathcal{M}^\nu_\alpha((-\infty, \delta])$ are Artin stacks of finite type.

Proof. (i) By Lemma 4.14 it suffices to show that

$$ \mathrm{Obj}^{\geq -1}({}^p\mathcal{B}) \subset \mathrm{Obj}(\mathrm{Coh}^\dagger(X)) $$

is open. This can be proved in the same way as [49, Lemma 5.1]. An object $P \in \mathrm{Coh}^\dagger$ of rank 0 (resp. -1) is contained in ${}^p\mathcal{B}$ if and only if $\det(P) = 0$ (resp. $\det(P) \cong \mathcal{O}_X$) and $\mathcal{H}^{-1}(P)$ is torsion-free on $X \setminus W$. The openness is proved using a spectral sequence argument as in [46, Lemma 3.16].

(ii) We explain that $\mathcal{M}^\nu_\gamma(I) \subset \mathrm{Obj}(\mathcal{A}_{\leq 1})$ is open and that the family of objects in $\mathcal{M}^\nu_\gamma(I)$ is bounded, if I is bounded. It follows that $\mathcal{M}^\nu_\gamma(I)$ is an Artin stack of finite type [45, Lemma 3.4]. By Corollary 4.9, there are only finitely many effective decompositions of γ in N_1^{eff}. Boundedness of the family of objects in $\mathcal{M}^\nu_\gamma(I)$ then follows from Lemma 2.14, Proposition 2.15, Proposition 4.10, Lemma 4.12 and Proposition 4.13.

Openness can be obtained from arguments of Toda [45, 49] as follows. In [49] he considers Calabi–Yau 3-folds X containing a divisor isomorphic to $\mathbb{P}^2$, and the category of sheaves with at most 1-dimensional support outside of the divisor. He studies objects in the tilt of this category along a torsion pair and proves boundedness of the family of semistable objects [49, Proposition 5.2]. Openness is deduced from boundedness as in [45, Theorem 3.20] and the same proof can be used for ν-stability.

(iii) Suppose that $E \in \mathcal{M}^\nu_\alpha\big([\delta, +\infty]\big)$ (the other case is analogous) and without loss of generality $\delta < 0$. Consider the decomposition

$$ E_0 \to E \to E_1 $$

of E with respect to the torsion pair $(\mathcal{A}_0, \mathcal{A}_1)$. Let $\gamma \in N_1$ be the residue of α. Then, $E_1 \in \mathcal{M}^\nu_\gamma\big([\delta, +\infty)\big)$, so for any subobject $E' \to E_1$ in $\mathcal{A}$ we have either $\nu(E') \leq 0$, or

$$\begin{aligned}\nu(E') \leq d(E') = d(E_1) - d(E_1/E') &\leq d(E) - \ell(E_1)\delta \\ &\leq d(\alpha) - \ell(\alpha)\delta,\end{aligned}$$

thus $E_1 \in \mathcal{M}^\nu_\gamma\big([\delta, \max\{0, d(\alpha) - \ell(\alpha)\delta\}]\big)$ is bounded. In particular, there are only finitely many possibilities for $d(E_1)$ and hence finitely many possibilities for $d(E_0)$, so the family of possible E_0 is bounded by Lemma 4.12. Using Lemma 4.11 we conclude (iii). □

The next lemma will be useful in the combinatorical analysis of the wall-crossing formula. Let A be the nef class of condition $(\diamondsuit)$. The restriction $\iota^* A$ is numerically equivalent to a multiple of b, thus multiplication by A defines a map

$$A \cdot (-) \colon N_{\leq 1} \to N_0 \,.$$

Lemma 4.16 ([5, Proposition 7.1.(3)])**.** For any $\gamma \in N_1$ the image of the set

$$\{c \in N_0 \mid \mathcal{M}^\nu_{(\gamma, c)} \neq \emptyset\}$$

in the quotient

$$N_0/\mathbb{Z}(A \cdot \gamma)$$

is finite.

Proof. The proof is the same as in [5], using Proposition 4.15. □

4.7 Refined stability

Finally we introduce the last stability function that we'll need. This stability function ζ will be used for the BS/^{p}PT wall-crossing and is the analog of [5, Definition 8.1].

For $E \in \mathcal{A}_{\leq 1} \setminus \{0\}$ define the function

$$\zeta(E) = \Big(-\frac{r}{\ell(E)}, \nu(E) \Big) \in (-\infty, +\infty] \times (-\infty, +\infty] \,,$$

where as before $r \in \mathbb{Z}$ such that $\mathrm{ch}_1(E) = r\mathsf{w}$. If $E \in \mathcal{A}_0$ we set

$$\zeta(E) = (+\infty, +\infty) \,.$$

We give $(-\infty,+\infty] \times (-\infty,+\infty]$ the lexicographic order. For $x, y \in (-\infty,+\infty] \times (-\infty,+\infty]$ we write $[x, y]$ and $]x, y]$ for the set of all z with $x \leq z \leq y$ resp. $x < z \leq y$. Note that the first component

$$\zeta_1(E) = -\frac{r}{\ell(E)}$$

only depends on the class of $[E]$ in $N_1 = N_{\leq 1}/N_0$. For $\gamma \in N_1$ we will also write $\zeta_1(\gamma)$.

Proposition 4.17. The slope ζ defines a stability condition on $\mathcal{A}_{\leq 1}$.

Proof. The see-saw property is straightforward. To prove that $\mathcal{A}_{\leq 1}$ is ζ-Artinian, the same strategy as in [5, Proposition 8.2] can be employed: by Corollary 4.9 it is enough to show that $\mathcal{A}_{\leq 1}$ is ν-Artinian, which we did in Proposition 4.4. □

Given a subset $I \subset (-\infty,+\infty] \times (-\infty,+\infty]$ we follow the notation of Section 4.5, so e.g. $\mathcal{M}^{\zeta}(I)$ is the stack of $E \in \mathcal{A}_{\leq 1}$ such that all their ζ-HN-factors are contained in I. To apply the wall-crossing formula to ζ-wall-crossing we will need to prove that the stacks $\mathcal{M}^{\zeta}(I)$ are open and (locally) of finite type.

For this, we recall the linear function $L_\mu \colon N_0 \to \mathbb{R}$ defined by

$$L_\mu(c) = L_\mu(j\mathsf{b}, n) = 2n + j + \frac{j}{\mu a_0}.$$

A set $S \subset N_0$ is said to be L_μ-bounded if for each $M \in \mathbb{R}$,

$$\#\{c \in S \colon L_\mu(c) < M\} < \infty.$$

We say that a set of objects in $\mathcal{A}_{\leq 1}$ is L_μ-bounded if its image in N_0 is L_μ-bounded.

Lemma 4.18 ([5, Lemma 8.14])**.** Given $\mu > 0, \eta_1, \eta_2 \in \mathbb{R}$ and $\gamma \in N_1$, the sets

$$\mathcal{M}^{\nu}_{\gamma}\big([\eta_1, +\infty]\big) \cap \mathcal{M}^{\zeta}_{\gamma}\big(](-\infty, -\infty), (\mu, \eta_2)]\big)$$

and

$$\mathcal{M}^{\nu}_{\gamma}\big((-\infty, \eta_1]\big) \cap \mathcal{M}^{\zeta}_{\gamma}\big([(\mu, \eta_2), (+\infty, +\infty)]\big)$$

are L_μ-bounded.

Proof. See [5, Lemma 8.14]. □

Proposition 4.19. Let $I \subset (-\infty,+\infty] \times (-\infty,+\infty]$ be an interval, $\gamma \in N_1$ and $(\mu, \eta) \in \mathbb{R}_{>0} \times \mathbb{R}$.

(i) The stack $\mathcal{M}^{\zeta}(I) \subset \mathrm{Obj}(\mathcal{A}_{\leq 1})$ is an open substack locally of finite type.

(ii) The family of objects in $\mathcal{M}^\zeta_\gamma(\mu,\eta)$ is L_μ-bounded.

Proof. Given $E \in \mathcal{M}^\zeta_\gamma(\mu,\eta)$ we consider its decomposition with respect to the ν-HN-filtration

$$E_{\geq\eta} \to E \to E_{<\eta}.$$

Then, both $\gamma' = [E_{\geq\eta}] \in N_1^{\mathrm{eff}}$ and $\gamma - \gamma' \in N_1^{\mathrm{eff}}$, and we have

$$E_{\geq\eta} \in \mathcal{M}^\nu_{\gamma'}\big([\eta,+\infty]\big) \cap \mathcal{M}^\zeta_{\gamma'}\big(](-\infty,-\infty),(\mu,\eta)]\big)\,.$$

By Corollary 4.9 there are finitely many such γ', so by Lemma 4.18 the set of possibilities for $c_{E_{\geq\eta}}$ is L_μ-bounded. Similarly, the possibilities for $c_{E_{<\eta}}$ are also L_μ-bounded and (ii) immediately follows.

For (i), by [45, Theorem 3.20] it is again enough to show that the family of semistable sheaves in $\mathcal{M}^\zeta_{(\gamma,c)}(\mu,\eta)$ is bounded. But using the decomposition above we have $c_{E_{\geq\eta}} + c_{E_{<\eta}} = c$, so there is a finite number of possibilities for both $c_{E_{\geq\eta}}$ and $c_{E_{<\eta}}$. It then follows from Proposition 4.15 (iii) that the families of possible $E_{\geq\eta}, E_{<\eta}$ are both bounded. By Lemma 4.11 we conclude that $\mathcal{M}^\zeta_{(\gamma,c)}(\mu,\eta)$ is bounded. □

5 Bryan–Steinberg

In this section we introduce numerical invariants $\mathrm{BS}_{\beta,n}$ that naturally realize the quotient

$$\mathrm{BS}_\beta(q,Q) = \frac{\mathrm{PT}_\beta(q,Q)}{\mathrm{PT}_0(q,Q)}\,.$$

The equation will be a wall-crossing formula between BS and PT invariants. When X admits a contraction map $X \to Y$ as in Section 1.2 these invariants are precisely Bryan–Steinberg invariants [11] of the crepant resolution. Roughly speaking they count a modification of pairs $\mathcal{O}_X \to F$ where instead of requiring the cokernel to have dimension zero we allow it to have support in some of the fibers B.

We define BS-pairs using a torsion pair of $\mathrm{Coh}_{\leq 1}(X)$. Let

$$\mathcal{T}_{\mathrm{BS}} = \big\{T \in \mathrm{Coh}_{\leq 1}(X)\colon\ T_{|X\setminus W} \in \mathrm{Coh}_0(X\setminus W) \text{ and } Rp_*\iota^*T \in \mathrm{Coh}_0(C)\big\}\,.$$

One easily checks that $\mathcal{T}_{\mathrm{BS}}$ is closed under quotients and extensions (see [11, Lemma 13] for the case where a contraction exists). In fact, $\mathcal{T}_{\mathrm{BS}}$ coincides with a previously defined subcategory:

$$\mathcal{T}_{\mathrm{BS}} = \mathcal{T}_0 = \mathcal{A}_0 \cap \mathcal{T}\,.$$

Then, the orthogonal complement

$$\mathcal{F}_{\mathrm{BS}} = \{F \in \mathrm{Coh}_{\leq 1}(X) : \mathrm{Hom}(\mathcal{T}_{\mathrm{BS}}, F) = 0\}$$

defines the torsion-free part of a torsion pair $(\mathcal{T}_{\mathrm{BS}}, \mathcal{F}_{\mathrm{BS}})$ of $\mathrm{Coh}_{\leq 1}(X)$.

The same proof as given in [11, Lemma 51] can be used to write the torsion pair $(\mathcal{T}_{\mathrm{BS}}, \mathcal{F}_{\mathrm{BS}})$ in terms of the stability condition μ^A introduced in Section 4.1:

$$\mathcal{T}_{\mathrm{BS}} = \mathcal{M}^{\mu^A}\Big(\big[\frac{\infty}{2}, +\infty\big[\Big)\,, \quad \mathcal{F}_{\mathrm{BS}} = \mathcal{M}^{\mu^A}\Big(\big] - \infty, \frac{\infty}{2}\big[\Big)\,,$$

where we used $\frac{\infty}{2}$ to denote

$$\frac{\infty}{2} = (+\infty, 0) \in (-\infty, +\infty] \times (-\infty, +\infty]\,.$$

The BS numerical invariants are defined as usual via the integration map I. We denote by $\mathrm{Pairs}^{\mathrm{BS}}$ the stack of $(\mathcal{T}_{\mathrm{BS}}, \mathcal{F}_{\mathrm{BS}})$-pairs in the sense of Definition 3.2. Then, we define $\mathrm{BS}_{\beta,n} \in \mathbb{Q}$ by the equation

$$I\big((\mathbb{L}-1)\mathrm{Pairs}^{\mathrm{BS}}\big) = \sum_{n,\beta} \mathrm{BS}_{n,\beta}\, z^\beta\, (-q)^n\, t^{-1}\,.$$

We also denote

$$\mathrm{BS}_\beta(q, Q) = \sum_{n,j \in \mathbb{Z}} \mathrm{BS}_{\beta + j\mathsf{b}, n}\, (-q)^n\, Q^j \in \mathbb{Q}[[q^{\pm 1}, Q^{\pm 1}]]\,.$$

5.1 Wall-crossing between BS and PT

The wall-crossing between BS and PT invariants can be directly deduced from the discussion in Section 3.4. Recall that the usual stable pairs are defined as pairs with respect to the torsion pair

$$(\mathcal{T}_{\mathrm{PT}}, \mathcal{F}_{\mathrm{PT}}) = \big(\mathrm{Coh}_0(X), \mathrm{Coh}_1(X)\big).$$

The technical conditions required in Section 3.4 are satisfied.

Proposition 5.1. The moduli $\mathrm{Pairs}^{\mathrm{BS}}_{(\beta,n)} \subset \mathcal{M}$ is an open substack of finite type. Moreover, the pairs $(\mathcal{T}_{\mathrm{PT}}, \mathcal{F}_{\mathrm{PT}})$ and $(\mathcal{T}_{\mathrm{BS}}, \mathcal{F}_{\mathrm{BS}})$ are wall-crossing material.

Proof. The torsion pair $(\mathcal{T}_{\mathrm{PT}}, \mathcal{F}_{\mathrm{PT}})$ is clearly open. The torsion pair $(\mathcal{T}_{\mathrm{BS}}, \mathcal{F}_{\mathrm{BS}})$ is also open thanks to the description in terms of μ^A stability and Proposition 4.1. By [5, Proposition 4.6] it follows that $\mathrm{Pairs}^{\mathrm{BS}}$, $\mathrm{Pairs}(\mathcal{T}_{\mathrm{PT}}, \mathcal{F}_{\mathrm{BS}})$ are open, locally of finite type substacks of $\mathcal{M}$.

To show that the pairs are wall-crossing material remains to show that $\mathcal{W} = \mathcal{F}_{\mathrm{PT}} \cap \mathcal{T}_{\mathrm{BS}}$ satisfies conditions (i)-(iii) in Section 3.4. Conditions (i) and (ii) are straightforward. For (iii), write $\alpha_i = (\beta_i, n_i)$. If $\mathcal{W}_{\alpha_i} \neq \emptyset$ we must have $\beta_i = j_i\mathsf{b}$ for some $j_i \geq 1$ and $n_i \geq 0$, so it is clear that there are only finitely many such decompositions. $\square$

Joyce's wall-crossing in Section 3.4 (or [5, Theorem 6.10]) now applies to show that for every $\beta \in N_1(X)$

$$\mathrm{PT}_\beta(q, Q) = f(q, Q)\,\mathrm{BS}_\beta(q, Q)\,, \tag{2.2}$$

where $f(q, Q)$ is defined by

$$f(q, Q) = I\big((\mathbb{L} - 1)\log([\mathcal{W}])\big) \in \mathbb{Q}[[q, Q]]$$

and $\mathcal{W} = \mathcal{F}_{\mathrm{PT}} \cap \mathcal{T}_{\mathrm{BS}} = \mathrm{Coh}_1(X) \cap \mathcal{T}_{\mathrm{BS}}$. Note that $f \in \mathbb{Q}[[q, Q]]$ because the support of sheaves in $\mathcal{W} \subset \mathcal{T}_{\mathrm{BS}}$ is a finite union of finitely many points and fibers B. Note also that f doesn't depend on β, so we get the relation

$$\frac{\mathrm{PT}_\beta(q, Q)}{\mathrm{BS}_\beta(q, Q)} = \frac{\mathrm{PT}_0(q, Q)}{\mathrm{BS}_0(q, Q)}\,.$$

Lemma 5.2. The only BS-pair of class $(-1, 0, j\mathsf{b}, n)$ is the trivial pair $(\mathcal{O}_X \to 0)$. In particular

$$\mathrm{BS}_0(q, Q) = 1\,.$$

Proof. The hypothesis of [5, Lemma 3.11], [46, Lemma 3.11 (ii)] applies to $\mathcal{T}_{\mathrm{BS}}$, showing that BS-pairs have the form $(\mathcal{O}_X \xrightarrow{s} G)$ where $G \in \mathcal{F}_{\mathrm{BS}}$ and $\mathrm{coker}(s) \in \mathcal{T}_{\mathrm{BS}}$. Since $\mathrm{Coh}_0(X) \subset \mathcal{T}_{\mathrm{BS}}$ we have $\mathcal{F}_{\mathrm{BS}} \subset \mathrm{Coh}_1(X)$, so G is a pure 1-dimensional sheaf. Since $\mathrm{ch}_2(G) = j\mathsf{b}$, the reduced support of G is a finite union of fibers B. Let Z be the subscheme of X determined by $\mathrm{Ker}(s) = I_Z$. We have an inclusion $\mathcal{O}_Z \hookrightarrow G$. The closed subspace underlying Z is a union of fibers B, so one easily sees that $\mathcal{O}_Z \in \mathcal{T}_{\mathrm{BS}}$. As $G \in \mathcal{F}_{\mathrm{BS}}$ it follows that $G = 0$. $\square$

As a consequence we get the key result of this section:

Proposition 5.3. We have

$$\mathrm{BS}_\beta(q, Q) = \frac{\mathrm{PT}_\beta(q, Q)}{\mathrm{PT}_0(q, Q)}\,.$$

We recall that $\mathrm{PT}_0(q, Q)$ can be computed (for example by localization on K_W, see appendix 8) and is equal to

$$\mathrm{PT}_0(q, Q) = \prod_{j \geq 1} (1 - q^j Q)^{(2g-2)j}\,.$$

6 Perverse PT invariants

Consider the torsion pair $(\mathcal{A}_0, \mathcal{A}_1)$ of $\mathcal{A}_{\leq 1}$ and recall the category

$$ {}^p\mathcal{B} = \Big\langle \mathcal{O}_X[1]\,, \mathcal{A}_{\leq 1} \Big\rangle \,. $$

An object $P \in {}^p\mathcal{B}$ is called *perverse stable pair*, if it is a $(\mathcal{A}_0, \mathcal{A}_1)$-pair in the sense of Definition 3.2, i.e. $\mathrm{rk}(P) = -1$ and

$$ \mathrm{Hom}(\mathcal{A}_0, P) = 0 = \mathrm{Hom}(P, \mathcal{A}_1)\,. $$

The stack of perverse pairs is denoted by ${}^p\mathrm{Pairs}$. Numerical invariants counting perverse stable pairs are defined using the integration map I as explained in Section 3. For $\alpha \in N_{\leq 1}$ we let ${}^p\mathrm{PT}_\alpha \in \mathbb{Q}$ be the numerical invariants defined by

$$ I\big((\mathbb{L}-1)\,{}^p\mathrm{PT}\big) = \sum_{(\gamma,j,n)} {}^p\mathrm{PT}_{(\gamma,j,n)}\, z^\gamma\, (-q)^n\, Q^j\, t^{-1}\,. $$

The fact that the integration map I can be applied to $(\mathbb{L}-1)\,{}^p\mathrm{PT}$ is justified by Lemmas 6.1 and 6.3.

In this section, we will provide a proof of the rationality and functional equation of perverse stable pairs, Theorem 1.2.

6.1 Rationality via ν-wall-crossing

For $\delta \in \mathbb{R}$ we introduce the torsion pair $(\mathcal{T}_{\nu,\delta}, \mathcal{F}_{\nu,\delta})$ on $\mathcal{A}_{\leq 1}$ by truncating the ν-HN-filtation at δ:

$$ \begin{aligned} \mathcal{T}_{\nu,\delta} &= \mathcal{M}^\nu\big([\delta, +\infty]\big) = \{E \in \mathcal{A}_{\leq 1} : E \twoheadrightarrow Q \neq 0 \Rightarrow \nu(Q) \geq \delta\}\ , \\ \mathcal{F}_{\nu,\delta} &= \mathcal{M}^\nu\big((-\infty, \delta)\big) = \{E \in \mathcal{A}_{\leq 1} : 0 \neq S \hookrightarrow E \Rightarrow \nu(S) < \delta\}\ . \end{aligned} $$

Here, $E \twoheadrightarrow Q$ and $S \hookrightarrow E$ means quotient resp. subobject in $\mathcal{A}_{\leq 1}$. This family of torsion pairs depending on δ will describe the wall-crossing which connects ${}^p\mathrm{Pairs}$ $(\delta \to +\infty)$ and $\rho({}^p\mathrm{Pairs})$ $(\delta \to -\infty)$. We denote by $\mathrm{Pairs}^{\nu,\delta}$ the stack of $(\mathcal{T}_{\nu,\delta}, \mathcal{F}_{\nu,\delta})$-pairs as defined in Section 3.3. This stack admits a decomposition into connected components according to the numerical class and we write $\mathrm{Pairs}^{\nu,\delta}_{(\gamma,c)}$ for the stack of pairs in class $(-1, \gamma, c)$.

Lemma 6.1. Let $\delta \in \mathbb{R}$ and $(\gamma, c) \in N_{\leq 1}$. The stack $\mathrm{Pairs}^{\nu,\delta}_{(\gamma,c)}$ is a finite type open substack of $\mathrm{Obj}^{\geq -1}({}^p\mathcal{B})$.

Proof. An object $P \in \mathrm{Obj}^{\geq -1}({}^p\mathcal{B})$ is a $(\mathcal{T}_{\nu,\delta}, \mathcal{F}_{\nu,\delta})$-pair if and only if three conditions hold:

(i) $\mathcal{H}^0(P) \in \mathcal{T}_{\nu,\delta}$,

(ii) $\mathcal{H}^0\big(\rho(P)\big) \in \big\langle \mathcal{A}_0\,, \rho(\mathcal{F}_{\nu,\delta})\big\rangle$,

(iii) $\mathcal{H}^1\big(\rho(P)\big) = 0$.

This characterization is parallel to the description of stable pairs (with respect to torsion theories) in $\big\langle \mathcal{O}_X[1]\,, \mathrm{Coh}_{\leq 1}(X)\big\rangle$ using the dualizing functor [5, Lemma 4.5]. Instead of the dualizing functor, we use the duality ρ and apply the same proof as [5, Proposition 4.6]. The necessary properties of ρ are proven in Section 2. The first and third properties are open by [5, Lemma 4.1], the second one by Theorem 2.5 and Property $(*)$. □

Applying the integration morphism in the Hall algebra produces numerical invariants ${}^p\mathrm{DT}^{\nu,\delta}_{(\gamma,c)} \in \mathbb{Q}$ defined by

$$I\big((\mathbb{L}-1)\mathrm{Pairs}^{\nu,\delta}\big) = \sum_{(\gamma,j,n)} {}^p\mathrm{DT}^{\nu,\delta}_{(\gamma,j,n)}\, z^{\gamma}\, (-q)^n\, Q^j\, t^{-1}\,. \tag{2.3}$$

Lemma 6.2 ([5, Proposition 7.6.(1)])**.** For any $\delta \in \mathbb{R}$ and $\gamma \in N_1$ the set

$$\{c \in N_0 : \mathrm{Pairs}^{\nu,\delta}_{(\gamma,c)} \neq 0\}$$

is finite.

Proof. The proof is an easy adaptation of the proof of [5, Proposition 7.6.(1)]. □

In the limit $\delta \to +\infty$ these invariants agree with the perverse PT invariants previously defined.

Lemma 6.3. Let $P \in {}^p\mathcal{B}$ be an object of class $(-1,\gamma,c)$. For $\delta \gg 0$ (depending on γ, c) we have

$$P \in \mathrm{Pairs}^{\nu,\delta} \text{ if and only if } P \in {}^p\mathrm{Pairs}.$$

Proof. The proof is analogous to [5, Lemma 7.10]. □

We will now apply Joyce's wall-crossing formula discussed in Section 3.4. The next lemma states the technical conditions under which we can use the wall-crossing formula.

Lemma 6.4. Let $\varepsilon > 0$ be sufficiently small. Then the torsion pairs $(\mathcal{T}_{\nu,\delta\pm\varepsilon}, \mathcal{F}_{\nu,\delta\pm\varepsilon})$ are wall-crossing material.

Proof. We begin by clarifying the statement and what we mean by sufficiently small ε. Fixing $l > 0$, the moduli of semistable sheaves $\mathcal{M}^{\nu}_{\leq l}(\delta')$ with $\ell(E) \leq l$ is empty unless $\delta' \in W_l = \frac{1}{l!}\mathbb{Z}$. Hence, by picking sufficiently small ε (depending on l) the intersection

$$\mathcal{W} = \mathcal{T}_{\nu,\delta-\varepsilon} \cap \mathcal{F}_{\nu,\delta+\varepsilon}$$

restricted to objects with $\ell(E) \leq l$ will be precisely $\mathcal{M}^{\nu}_{\leq l}(\delta)$. This will suffice for the way we'll write the wall-crossing formula.

Now for the actual proof. The stacks of pairs $\mathrm{Pairs}^{\nu,\delta\pm\varepsilon}$ define elements in the (graded pre-)Hall algebra by Lemma 6.1. It is then enough to show that $\mathcal{W} = \mathcal{M}^{\nu}(\delta)$ satisfies conditions (i)-(iii) of Section 3.4. Condition (ii) is obvious and condition (i) is proven in Proposition 4.15. For (iii), let $\alpha_i = (\gamma_i, c_i)$. By Corollary 4.9 there are finitely many possibilities for each γ_i. It also follows from Proposition 4.15 that for fixed δ, γ_i there are only finitely many c_i so that $\mathcal{M}^{\nu}_{(\gamma_i,c_i)}(\delta)$ is non-empty. □

By the previous lemma, we can define the invariants J^{ν}_{α} for $\alpha \in N_{\leq 1}$ by counting semistable perverse sheaves with respect to the slope ν:

$$I\big((\mathbb{L}-1)\log\left(\mathcal{M}^{\nu}(\delta)\right)\big) = \sum_{\nu(\alpha)=\delta} J^{\nu}_{\alpha}\, z^{\alpha}. \tag{2.4}$$

The J-invariants are analogous to Toda's N-invariants in the proof of the rationality of stable pairs generating functions.

The wall-crossing formula between ${}^{p}\mathrm{PT}$ and ${}^{p}\mathrm{DT}^{\nu,\delta_0}$ is

$${}^{p}\mathrm{PT}_{\leq l}\, t^{-1} = \left(\prod_{\delta \in W_l \cap [\delta_0,+\infty)} \exp\left(\{J_{\leq l}(\delta), -\}\right)\right) {}^{p}\mathrm{DT}^{\nu,\delta_0}_{\leq l}\, t^{-1}. \tag{2.5}$$

Here the subscript $\leq l$ means we're restricting the generating functions to the classes $\alpha \in N_{\leq 1}$ such that $\ell(\alpha) \leq l$. Moreover,

$$W_l = \frac{1}{l!}\mathbb{Z}$$

is the set of possible walls since $\ell(\alpha) \leq l$ implies $\nu(\alpha) \in W_l$.

Remark 6.5. In the wall-crossing formula (2.5) the wall-crossing terms interact, i.e. $\{J(\delta), J(\delta')\}$ might be non-trivial. In the usual proof of rationality of PT generating series or in the BS/PT wall-crossing this phenomenom doesn't happen because the wall-crossing terms are at most 1-dimensional, and χ vanishes when restricted to $\mathrm{Coh}_{\leq 1} \times \mathrm{Coh}_{\leq 1}$. However, that's no longer the case in $\mathcal{A}_{\leq 1} \times \mathcal{A}_{\leq 1}$ due to the presence of surface-like objects. In particular we don't get a product formula for wall-crossing similar to Proposition 5.3. The same phenomenon already happens in [5].

6.2 Combinatorics of the wall-crossing formula

Expanding the right-hand side of the wall-crossing formula (2.5) and extracting the coefficient of $z^\gamma t^{-1}$ we get the following expression for the perverse PT invariants in class $\gamma \in N_1$. The generating series

$$ {}^p\mathrm{PT}_\gamma = \sum_{j,n} {}^p\mathrm{PT}_{(\gamma,j,n)} (-q)^n Q^j = \sum \dots \tag{2.6} $$

is a sum over a set of choices described by an integer $m \in \mathbb{Z}_{\geq 0}$ and classes $\alpha_1, \dots, \alpha_i = (\gamma_i, c_i), \dots, \alpha_m \in N_{\leq 1}$ and $\alpha' = (\gamma', c') \in N_{\leq 1}$, satisfying the following conditions:

(i) $\gamma = \gamma' + \sum_{i=1}^m \gamma_i$,

(ii) $\delta_0 \leq \nu(\alpha_1) \leq \dots \leq \nu(\alpha_m)$,

(iii) $J^\nu_{\alpha_i} \neq \emptyset$,

(iv) ${}^p\mathrm{DT}^{\delta,\nu}_{\alpha'} \neq \emptyset$.

We now use the boundedness results to analyze this sum. First, conditions (iii) and (iv) imply that $\gamma_i, \gamma' \in N_1^{\mathrm{eff}}$. Together with condition (i) and Corollary 4.9 it follows that there is only a finite number of possibilities for γ_i, γ'. Lemma 6.2 also tells us that there is only a finite number of possibilities for α'. Finally, Lemma 4.16 says that after we fix $\gamma_1, \dots, \gamma_m$ there are finitely many possibilities for the classes $\kappa_i = [c_i] \in N_0/\mathbb{Z}(A \cdot \gamma_i)$. Since twisting by $\mathcal{O}_X(A)$ induces an isomorphism

$$ \mathcal{M}^\nu_{(\gamma_i, c_i)} \cong \mathcal{M}^\nu_{(\gamma_i, c_i + A\cdot\gamma_i)} , $$

it follows that $J^\nu_{(\gamma_i,c_i)}$ depends only on γ_i and the class $\kappa_i = [c_i]$, so we write $J^\nu_{(\gamma_i,\kappa_i)} = J^\nu_{(\gamma_i,c_i)}$.

Due to the combinatorical factor in (2.6) we also introduce the set $\mathcal{J}$ tracking which of the inequalities in (ii) are strict:

$$ \mathcal{J} = \{ i \in \{1, \dots, m-1\} : \nu(\alpha_i) = \nu(\alpha_{i+1}) \} . $$

We group the terms in the right hand side of (2.6) in finitely many groups according to the data $\xi = (\{\gamma_i\}_i, \{\kappa_i\}_i, \gamma', c', \mathcal{J})$. Since

$$ \nu(\gamma_i, c_i + A \cdot \gamma_i) = \nu(\gamma_i, c_i) + 1 , $$

given a group ξ we can chose a minimal set of representatives $c_i^0 \in \kappa_i$ such that

$$ \delta_0 \leq \nu(\gamma_1, c_1^0) < \delta_0 + 1 , \quad \nu(\gamma_i, c_i^0) \leq \nu(\gamma_{i+1}, c_{i+1}^0) < \nu(\gamma_i, c_i^0) + 1 . $$

Then we organize equation (2.6) as

$$
{}^{p}\mathrm{PT}_{\gamma} = \sum_{\xi} A(\xi) \sum_{(k_1,\ldots,k_m)\in S_{\mathcal{J}}} B_{\xi}(k_1,\ldots,k_m)\, z^{c'+\sum_{i=1}^{m}(c_i^0+k_i(A\cdot\gamma_i))} \tag{2.7}
$$

where the first sum runs over the finitely many possible groups and the second sum runs over the set

$$
S_{\mathcal{J}} = \left\{(k_1 \le \ldots \le k_m) : k_i = k_{i+1} \Leftrightarrow i \in \mathcal{J}\right\}.
$$

Since B_{ξ} is a quasi-polynomial of period 2, the rationality of ${}^{p}\mathrm{PT}_{\gamma}$ follows from [5, Lemma 2.21].

6.3 Functional equation

After we have established the rationality part of Theorem 1.2, we turn to the functional equation. For this, the duality ρ introduced in Section 2.3 plays a crucial role.

Lemma 6.6. Let $\delta \in \mathbb{R} \setminus \mathbb{Q}$. Then

$$
\rho\big(\mathrm{Pairs}^{\nu,\delta}\big) = \mathrm{Pairs}^{\nu,-\delta}.
$$

In particular,

$$
{}^{p}\mathrm{DT}_{\alpha}^{\nu,\delta} = {}^{p}\mathrm{DT}_{\rho(\alpha)}^{\nu,-\delta}.
$$

Proof. The lemma is proven exactly as in [5, Lemma 7.4], replacing $\mathrm{Coh}(\mathcal{Y})$ by $\mathcal{A}$ and $\mathbb{D}^{\mathcal{Y}}$ by ρ. The properties of ρ needed for the proof are Theorem 2.2 and Proposition 4.5. □

Lemma 6.7. Let $\gamma \in N_1$. We have

$$
\lim_{\delta\to-\infty} \deg\left({}^{p}\mathrm{PT}_{\gamma} - {}^{p}\mathrm{DT}_{\gamma}^{\nu,\delta}\right) = -\infty.
$$

Proof. We consider the wall-crossing equation (2.7) with $\delta = \delta_0$. Note that the terms in (2.7) with $m = 0$ (that is, in groups $\xi = (\emptyset, \emptyset, \gamma, c, \emptyset)$) give precisely ${}^{p}\mathrm{DT}_{\gamma}^{\nu,\delta}$, so we may express the difference ${}^{p}\mathrm{PT}_{\gamma} - {}^{p}\mathrm{DT}_{\gamma}^{\nu,\delta}$ as the sum on the right-hand side of (2.7) restricted to $m \ge 1$. Thus we have

$$
\deg\left({}^{p}\mathrm{PT}_{\gamma} - {}^{p}\mathrm{DT}_{\gamma}^{\nu,\delta}\right) \le \max_{\xi}\left(d(c') + \sum_{i=1}^{m} d(c_i^0)\right)
$$

where the max is taken over the groups ξ with $m \geq 1$. Summing $d(\gamma)$ to both sides

$$\deg\left({}^p\mathrm{PT}_\gamma - {}^p\mathrm{DT}^{\nu,\delta}_\gamma\right) + d(\gamma) \leq \max_\xi \left(d(\gamma', c') + \sum_{i=1}^m d(\gamma_i, c_i^0) \right) .$$

By the minimality of c_i^0 we know that $d(\gamma_i, c_i^0) < \delta_0 + i$, and therefore we get the bound

$$\deg\left({}^p\mathrm{PT}_\gamma - {}^p\mathrm{DT}^{\nu,\delta}_\gamma\right) + d(\gamma) \leq \max_\xi \left(d(\gamma', c') + m\delta_0 + \frac{m(m+1)}{2} \right) .$$

Now taking $\delta \to -\infty$ gives the desired limit. □

By Lemmas 6.3 and 6.6, for any $\alpha \in N_{\leq 1}$ and sufficiently small δ we have ${}^p\mathrm{DT}^{\nu,\delta}_\alpha = {}^p\mathrm{PT}_{\rho(\alpha)}$. Thus, we have

$${}^p\mathrm{DT}^{\nu,-\infty}_\alpha = \lim_{\delta\to-\infty} {}^p\mathrm{DT}^{\nu,\delta}_\alpha = \lim_{\delta\to+\infty} {}^p\mathrm{DT}^{\nu,\delta}_{\rho(\alpha)} = {}^p\mathrm{PT}_{\rho(\alpha)} .$$

Here, $\rho(\alpha)$ denotes the action on cohomology induced by ρ determined by Proposition 2.6. One can write this action as $\rho(\gamma, c) = \big(\gamma, \rho_\gamma(c)\big)$, where for each $\gamma = (r\mathsf{w}, \beta)$ the involution $\rho_\gamma \colon N_0 \to N_0$ is

$$\rho_\gamma(j\mathsf{b}, n) = \big((-j + \mathsf{w}\cdot\beta - (2-2g)r)\mathsf{b}, -n\big) .$$

We write the previous relation between ${}^p\mathrm{DT}^{\nu,-\infty}$ and ${}^p\mathrm{PT}$ as an equality of generating functions for $\gamma \in N_1$:

$${}^p\mathrm{DT}^{\nu,-\infty}_\gamma = \sum_{c\in N_0} {}^p\mathrm{DT}^{\nu,-\infty}_{(\gamma,c)} z^c = \sum_{c\in N_0} {}^p\mathrm{PT}_{(\gamma,\rho_\gamma(c))} z^c = \rho_\gamma({}^p\mathrm{PT}_\gamma) .$$

It follows that ${}^p\mathrm{DT}^{\nu,-\infty}_\gamma$ is the expansion of a rational function in $\mathbb{Q}[q,Q]_{-d}$. On the other hand, by Lemma 6.3

$$\lim_{\delta\to-\infty} \deg\left({}^p\mathrm{DT}^{\nu,-\infty}_\gamma - {}^p\mathrm{DT}^{\nu,\delta}_\gamma\right) = -\infty$$

and, together with Lemma 6.7, we have an equality of rational functions

$${}^p\mathrm{PT}_\gamma = {}^p\mathrm{DT}^{\nu,-\infty}_\gamma = \rho_\gamma\left({}^p\mathrm{PT}_\gamma\right) .$$

This finishes the proof of Theorem 1.2.

7 Bryan–Steinberg vs. perverse PT invariants

In this section we will prove the wall-crossing between the Bryan–Steinberg invariants and perverse PT invariants. Together with the BS/PT wall-crossing of Section 5, the output of this section is a proof of Theorem 1.3.

We will use the stability condition ζ defined in Section 4.7. The wall-crossing is entirely analogous to [5, Section 8], where Bryan–Steinberg pairs are compared to orbifold PT pairs to prove the crepant resolution conjecture. For us, matters simplify and it is worth to point out how exactly.

The stability ζ leads to torsion pairs $(\mathcal{T}_{\zeta,(\mu,\eta)}, \mathcal{F}_{\zeta,(\mu,\eta)})$ on $\mathcal{A}_{\leq 1}$ labelled by $(\mu,\eta) \in \mathbb{R}_{>0} \times \mathbb{R}$. These are defined analogously to $(\mathcal{T}_{\nu,\delta}, \mathcal{F}_{\nu,\delta})$ in Section 6.1, by truncating the ζ-HN-filtration. We consider the stack $\mathrm{Pairs}^{\zeta,(\mu,\eta)}$ of $(\mathcal{T}_{\zeta,(\mu,\eta)}, \mathcal{F}_{\zeta,(\mu,\eta)})$ pairs in

$$ {}^p\mathcal{B} = \left\langle \mathcal{O}_X[1], \mathcal{A}_{\leq 1} \right\rangle $$

in the sense of Definition 3.2.

Lemma 7.1. Let $(\mu,\eta) \in \mathbb{R}_{>0} \times \mathbb{R}$ and $(\gamma, c) \in N_{\leq 1}$.

(i) The stack $\mathrm{Pairs}^{\zeta,(\mu,\eta)}_{(\gamma,c)} \subset \mathrm{Obj}^{\geq -1}({}^p\mathcal{B})$ is an open substack of finite type.

(ii) The family of objects in $\mathrm{Pairs}^{\zeta,(\mu,\eta)}_{\gamma}$ is L_μ-bounded.

Proof. The same strategy of [5, Proposition 8.16] can be employed to prove the result from Lemmas 4.18 and 6.1 and Proposition 4.19. □

We define numerical invariants

$$ {}^p\mathrm{DT}^{\zeta,(\mu,\eta)}_{\gamma,c} \in \mathbb{Q} $$

as we did for pairs defined using ν in Section 6, see equation (2.3).

The notion of (μ,η)-pairs is locally constant. More precisely, for fixed $\gamma \in N_1$ there is a finite set of possible walls V_γ such that stability is constant on

$$ \left(\mathbb{R}_{>0} \setminus V_\gamma\right) \times \mathbb{R}\,. $$

The limit $0 < \mu \ll 1$ coincides with BS-pairs, the limit $\mu \to +\infty$ coincides with perverse stable pairs. Crossing a wall $\mu \in V_\gamma$ leads to a wall-crossing formula. This wall-crossing is controlled in a concrete way. There is precisely one effective class $0 < \gamma' \leq \gamma$ characterized by $L_\mu(A \cdot \gamma') = 0$, where as before

$$ L_\mu(j,n) = 2n + j + \frac{j}{\mu\, a_0}\,. $$

The asymmetry of n and j in this formula hints at how varying μ separates BS from perverse PT (see Example 7.2 below). Recall that L_μ is the same linear function introduced in Section 3 that controls the expansion of the rational function.

Then, to cross the μ-wall, it is possible to enter the wall from either sides because for $0 < \varepsilon \ll 1$ we have

$$\mathrm{Pairs}^{\zeta,(\mu\pm\varepsilon,\eta)} = \mathrm{Pairs}^{\zeta,(\mu,\pm\infty)}\,.$$

The wall-crossing inside $\{\mu\} \times \mathbb{R}$ is similar to the ν-wall-crossing in Section 6. The combinatorics is controlled in the same way.

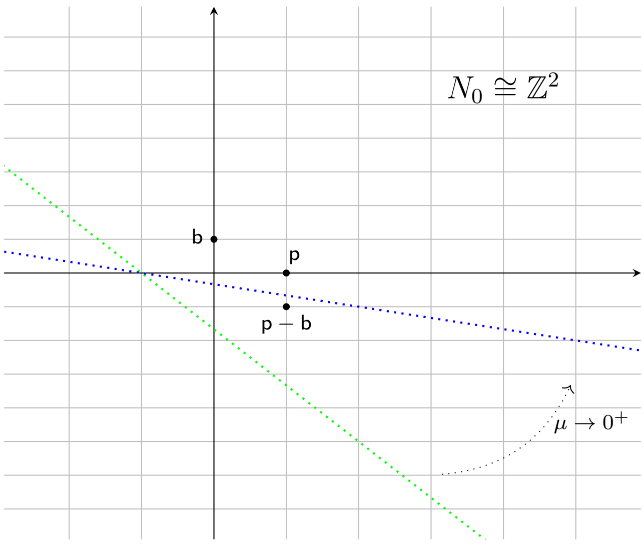
Example 7.2. We include an illustration of the wall-crossing for the limit $\mu \to 0^+$. Let $B \subset W$ be a $\mathbb{P}^1$-fiber of the projection. Since $\chi\big(\mathcal{O}_B(-1)\big) = 0$, the class b of the ruling is identified with the K-theory class $[\mathcal{O}_B(-1)]$. The linear function L_μ specifies which classes in N_0 are considered effective. Recall the structure sheaves $k(x)$ of points in X and the perverse sheaves $\mathcal{O}_B(-1)$ and $\mathcal{O}_B(-2)[1]$ in $\mathcal{A}_0$. Their K-theory classes are

$$\big[k(x)\big] = \mathsf{p}\,, \quad \big[\mathcal{O}_B(-1)\big] = \mathsf{b}\,, \quad \big[\mathcal{O}_B(-2)[1]\big] = \mathsf{p} - \mathsf{b}\,.$$

Both p and b satisfy $L_\mu > 0$ for all $\mu > 0$, i.e. both classes are considered effective at all times. In contrast to that, the class of $\mathcal{O}_B(-2)[1]$ (considered effective for perverse stable pairs) satisfies

$$\begin{aligned} L_\mu(\mathsf{p}-\mathsf{b}) &> 0\,, \qquad \mu > 1\,,\\ L_\mu(\mathsf{p}-\mathsf{b}) &< 0\,, \qquad 0 < \mu \ll 1\,. \end{aligned}$$

The limit $\mu \to 0^+$ serves the purpose to exclude all such perverse sheaves (two-term complexes in $\mathcal{A}_0$) from being considered effective.



The picture displays two lines $\{L_\mu = 0\}$ in $N_0 \cong \mathbb{Z}^2$. For $\mu \gg 1$ (green dotted line) the class $\mathsf{p} - \mathsf{b}$ is effective, i.e. contained in $\{L_\mu > 0\}$, for $0 < \mu \ll 1$ (blue dotted line) it is not.

7.1 Walls

Let $\gamma \in N_1$. Define the set of possible walls

$$V_\gamma = \{\zeta_1(\gamma') : 0 < \gamma' \leq \gamma\} \cap \mathbb{R}_{>0} .$$

The stack $\mathrm{Pairs}_\gamma^{\zeta,(\mu,\eta)}$ is constant on $(\mathbb{R}_{>0} \setminus V_\gamma) \times \mathbb{R}$.

In the following, when $\mu \in (\mathbb{R}_{>0} \setminus V_\gamma)$ we let $\eta \in \mathbb{R}$ arbitrary. Crossing a wall $\mu \in V_\gamma$ is controlled by the linear function L_μ. The basic reason is the following relation between L_μ and ζ_1:

$$L_\mu(A \cdot \gamma) = d(A \cdot \gamma) \left(1 - \frac{\zeta_1(\gamma)}{\mu}\right) .$$

Lemma 7.3. There is, up to scaling, a unique class γ_μ such that $0 < \gamma_\mu \leq \gamma$ and $L_\mu(A \cdot \gamma_\mu) = 0$. The class $A \cdot \gamma_\mu \in N_0$ is uniquely characterized by this property.

Proof. The proof is a simplified version of [5, Lemma 8.21].[8] □

Example 7.4. We illustrate the previous result for $W \cong \mathbb{P}^1 \times \mathbb{P}^1$ with projection $p \colon W \to \mathbb{P}^1$. Let B and C be a fiber resp. section of p and

$$\mathsf{b} = [B] , \quad \mathsf{c} = [C]$$

their classes in N_1. Consider the class $\gamma = \mathsf{c} - \mathsf{b} \in N_1$. It is an effective class:

$$\gamma = [\mathcal{O}_W(-2C - B)[1]] + [\mathcal{O}_W(-C - 2B)] .$$

The two objects are contained in $\mathcal{F}[1]$ and $\mathcal{T}_{\leq 1}$ respectively and the sum gives rise to the effective decomposition

$$\gamma = (-\mathsf{w}, \mathsf{c}) + (\mathsf{w}, -\mathsf{b}) .$$

Recall the line bundle A and $\ell(r, \beta) = 2A \cdot \beta + r\, a_0$. We have

$$\zeta_1(\mathcal{O}_W(-2C - B)[1]) = -\frac{-1}{2a_0 - a_0} = \frac{1}{a_0}$$

[8] In [5] the authors choose a very general ample class to define the stability ζ and function L_μ. This choice is not necessary for our application because $\mathrm{ch}_1(E) \in \mathbb{Z} \cdot \mathsf{w}$ for all $[E] \in N_{\leq 1}$ and $\mathrm{ch}_2(E) \in \mathbb{Z} \cdot \mathsf{b}$ for all $[E] \in N_0$.

and there is only one wall

$$V_\gamma = \left\{ \frac{1}{a_0} \right\}.$$

The unique class γ_μ is $\big[\mathcal{O}_W(-2C - B)[1]\big] = (-\mathsf{w}, \mathsf{c})$ and

$$A \cdot \gamma_\mu = (-a_0\,, a_0) \in N_0\,.$$

The linear function L_μ uniquely specifies $A \cdot \gamma_\mu$ as

$$L_{\mu'}(A \cdot \gamma_\mu) \begin{cases} > 0\,, & \mu' > \frac{1}{a_0}\,, \\ = 0\,, & \mu' = \frac{1}{a_0}\,, \\ < 0\,, & \mu' < \frac{1}{a_0}\,. \end{cases}$$

Correspondingly, the class $A \cdot \gamma_\mu = a_0\big[\mathcal{O}_B(-2)[1]\big] \in N_0$ is considered effective in the expansion of the rational function with respect to $L_{\mu'}$ for $\mu' > \frac{1}{a_0}$ (^{p}PT pairs), whereas it is non-effective for $\mu' < \frac{1}{a_0}$ (BS-pairs).

7.2 Limit stability I

We identify the limit of (μ, η)-stability for $0 < \mu \ll 1$ with BS stability. First, we can give an explicit description of the limit of the torsion pair for $0 < \mu \ll 1$.

Definition 7.5. We define the torsion pair $(\mathcal{T}_{\zeta,0}, \mathcal{F}_{\zeta,0})$ in $\mathcal{A}_{\leq 1}$ by

$$\mathcal{T}_{\zeta,0} = \big\{ E \in \mathcal{A}_{\leq 1} : E \twoheadrightarrow Q \Rightarrow Q \in \mathcal{A}_0 \text{ or } \mathrm{ch}_1(Q) \in \mathbb{Z}_{<0} \cdot \mathsf{w} \big\}$$

and the orthogonal complement $\mathcal{F}_{\zeta,0} = \mathcal{T}_{\zeta,0}^{\perp}$.

It is straightforward to see that the pair $(\mathcal{T}_{\zeta,0}, \mathcal{F}_{\zeta,0})$ is the limit of $(\mathcal{T}_{\zeta,(\mu,\eta)}, \mathcal{F}_{\zeta,(\mu,\eta)})$ when μ becomes very small, in the following precise sense:

Lemma 7.6. Let $P \in {}^p\mathcal{B}$ of class $(-1, \gamma, c)$ and $0 < \mu < \min V_\gamma$. Then, P is a $(\mathcal{T}_{\zeta,0}, \mathcal{F}_{\zeta,0})$ pair if and only if P is a $(\mathcal{T}_{\zeta,(\mu,\eta)}, \mathcal{F}_{\zeta,(\mu,\eta)})$ pair.

Lemma 7.7. We have

$$\mathcal{T}_{\zeta,0} = \big\langle \mathcal{F}[1], \mathcal{T}_0 \big\rangle_{\mathrm{ex}}$$
$$\mathcal{F}_{\zeta,0} = \mathcal{T}_1.$$

Proof. We begin by proving that $\big\langle \mathcal{F}[1], \mathcal{T}_0 \big\rangle_{\mathrm{ex}} \subset T_{\zeta,0}$. We first note that we can write

$$\mathcal{A}_{\leq 1} = \big\langle \mathcal{F}[1], \mathcal{T}_{\leq 1} \big\rangle = \big\langle \mathcal{F}[1], \mathcal{T}_0, \mathcal{T}_1 \big\rangle = \big\langle \langle \mathcal{F}[1], \mathcal{T}_0 \rangle_{\mathrm{ex}}, \mathcal{T}_1 \big\rangle\,,$$

so $\langle \mathcal{F}[1], \mathcal{T}_0 \rangle_{\text{ex}}$ is closed under quotients. Hence it is enough to show that if $E \in \mathcal{F}[1]$ or $E \in \mathcal{T}_0$ then $E \in \mathcal{A}_0$ or $\text{ch}_1(E) \in \mathbb{Z}_{<0} \cdot \mathsf{w}$. For $T \in \mathcal{T}_0$ this is clear. If $F[1] \in \mathcal{F}[1]$ then $\text{ch}_1(F[1]) = r\mathsf{w}$ with $r \leq 0$ and equality if and only if $F \in \text{Coh}_{\leq 1}(X)$. So it remains to show that if $F \in \mathcal{F}$ and $\text{ch}_1(F) = 0$ then $F \in \mathcal{F}_0$, i.e. $\mathcal{F} \cap \text{Coh}_{\leq 1}(X) = \mathcal{F}_0$.

We let $F \in \mathcal{F} \cap \text{Coh}_{\leq 1}(X)$ and, by Lemma 2.14, may assume F is scheme-theoretically supported on W. If there is a fiber $B = p^{-1}(y)$ such that $\text{supp}(F) \cap B$ is 0-dimensional and non-empty then $p_* F \otimes k(y) \neq 0$ contradicting Lemma 2.10 (i). Thus, $\text{supp}(F)$ is a finite union of fibers B, so $F \in \mathcal{F}_0$ proving the claim.

For the inclusion $T_{\zeta,0} \subset \langle \mathcal{F}[1], \mathcal{T}_0 \rangle_{\text{ex}}$, let $E \in T_{\zeta,0}$ and consider the decomposition of E with respect to the torsion pair $(\mathcal{F}[1], \mathcal{T}_{\leq 1})$ of $\mathcal{A}_{\leq 1}$:

$$0 \to F[1] \to E \to T \to 0.$$

Since $\text{ch}_1(T) \in \mathbb{Z}_{\geq 0} \cdot \mathsf{w}$, by the definition of $\mathcal{T}_{\zeta,0}$ we have $T \in \mathcal{T} \cap \mathcal{A}_0 = \mathcal{T}_0$.

This finishes the proof of the first equality $T_{\zeta,0} = \langle \mathcal{F}[1], \mathcal{T}_0 \rangle_{\text{ex}}$. The second equality follows from the first one by taking orthogonal complements in $\mathcal{A}_{\leq 1}$. ☐

Recall that $\mathcal{T}_0 = \mathcal{T} \cap \mathcal{A}_0 = \mathcal{T}_{\text{BS}}$, so in particular $\mathcal{T}_{\text{BS}} \subset \mathcal{T}_{\zeta,0}$. The key result of this section is:

Proposition 7.8. Let $P \in D^b(X)$ be such that $\text{ch}_1(P) = 0$. Then P is a $(\mathcal{T}_{\zeta,0}, \mathcal{F}_{\zeta,0})$-pair if and only if P is a $(\mathcal{T}_{\text{BS}}, \mathcal{F}_{\text{BS}})$-pair. In particular, for any $\beta \in N_1(X)$ and $0 < \mu < \min V_\beta$ we have

$$^p\text{DT}_\beta^{\zeta,(\mu,\eta)} = \text{BS}_\beta\,.$$

Proof. We begin with the proof that if P is a $(\mathcal{T}_{\text{BS}}, \mathcal{F}_{\text{BS}})$-pair then it is a $(\mathcal{T}_{\zeta,0}, \mathcal{F}_{\zeta,0})$-pair. If P is a BS-pair, by [5, Lemma 3.11], [46, Lemma 3.11 (ii)] we can write $P = (\mathcal{O}_X \xrightarrow{s} F)$ with

$$F \in \mathcal{F}_{\text{BS}}\,, \quad Q = \text{coker}(s) \in \mathcal{T}_{\text{BS}} = \mathcal{T}_0 \subset \mathcal{A}_{\leq 1}\,.$$

We first prove that $F \in \mathcal{A}_{\leq 1}$, so $P \in {}^p\mathcal{B}$. If Z is the scheme-theoretical support of F (which is a curve), we have the short exact sequence of sheaves

$$0 \to \mathcal{O}_Z \to F \to Q \to 0.$$

Since both $\mathcal{O}_Z$ and Q are contained in $\mathcal{A}_{\leq 1}$, which is closed under extensions, it follows that $F \in \mathcal{A}_{\leq 1}$. Moreover for $T \in \mathcal{T}_{\zeta,0}$

$$\text{Hom}(T, P) = \text{Hom}(T, F) = \text{Hom}(H^0(T), F) = 0\,.$$

The last vanishing holds because $H^0(T) \in \mathcal{T}_0 = \mathcal{T}_{\mathrm{BS}}$ by Lemma 7.7 and $F \in \mathcal{F}_{\mathrm{BS}}$. Similarly, for $G \in \mathcal{F}_{\zeta,0}$,

$$\mathrm{Hom}(P, G) = \mathrm{Hom}(Q, G) = 0$$

vanishes since $Q \in \mathcal{T}_{\mathrm{BS}} \subset \mathcal{T}_{\zeta,0}$. So we conclude that P is a $(\mathcal{T}_{\zeta,0}, \mathcal{F}_{\zeta,0})$-pair.

We now assume that P is a $(\mathcal{T}_{\zeta,0}, \mathcal{F}_{\zeta,0})$-pair with $\mathrm{ch}_1(P) = 0$. Since

$$P \in {}^p\mathcal{B} = \left\langle \mathcal{O}_X[1], \mathcal{F}[1], \mathcal{T}_{\leq 1} \right\rangle,$$

$\mathcal{H}^{-1}(P)$ has rank 1, $\mathcal{H}^0(P)$ has rank 0, and $\mathcal{H}^i(P) = 0$ for $i \neq -1, 0$. Moreover, the torsion part $T \hookrightarrow \mathcal{H}^{-1}(P)$ is in $\mathcal{F}$, so $T[1] \in \mathcal{F}[1] \subset \mathcal{T}_{\zeta,0}$. By definition of $(\mathcal{T}_{\zeta,0}, \mathcal{F}_{\zeta,0})$-pair the composition

$$T[1] \hookrightarrow \mathcal{H}^{-1}(P)[1] \to P$$

vanishes, forcing T to vanish. Thus $\mathcal{H}^{-1}(P)$ is torsion-free. By Lemma 7.7 we have

$$\mathcal{H}^0(P) \in \mathcal{T}_{\zeta,0} \cap \mathrm{Coh}(X) = \mathcal{T}_0 = \mathcal{T}_{\mathrm{BS}}.$$

In particular it follows that

$$\mathrm{ch}_1\big(\mathcal{H}^{-1}(P)\big) = \mathrm{ch}_1\big(\mathcal{H}^0(P)\big) - \mathrm{ch}_1(P) = 0.$$

Hence $\mathcal{H}^{-1}(P)$ is a torsion-free, rank 1 sheaf with trivial determinant, hence it is an ideal sheaf $\mathcal{H}^{-1}(P) \cong I_C$. So P fits in an exact triangle

$$I_C[1] \to P \to \mathcal{H}^0(P).$$

Using the argument of [46, Lemma 3.11 (ii)] with the fact that

$$H^1\big(X, \mathcal{H}^0(P)\big) = 0\,,$$

we get that P has the form $P = (\mathcal{O}_X \to F)$. We already know that $\mathcal{H}^0(P) \in \mathcal{T}_{\mathrm{BS}}$ so it remains to show that $F \in \mathcal{F}_{\mathrm{BS}}$ (see [5, Remark 3.10]). For $T \in \mathcal{T}_{\mathrm{BS}}$ we have

$$\mathrm{Hom}(T, F) = \mathrm{Hom}(T, P) = 0$$

since $T \in \mathcal{T}_{\mathrm{BS}} \subset \mathcal{T}_{\zeta,0}$, and we're done. □

7.3 Limit stability II

We identify the limit of (μ, η)-stability for $\mu \to \infty$ with ^{p}PT stability.

Lemma 7.9. Let $P \in {}^p\mathcal{B}$ be of class $(-1, \gamma, c)$ and $\mu > \max V_\gamma$. Then, P is a perverse stable pair if and only if P is a $(\mathcal{T}_{\zeta,(\mu,\eta)}, \mathcal{F}_{\zeta,(\mu,\eta)})$ pair. In particular, for any $\gamma \in N_1$ and $\mu > \max V_\gamma$ we have

$${}^p\mathrm{DT}_\gamma^{\zeta,(\mu,\eta)} = {}^p\mathrm{PT}_\gamma\,.$$

Proof. The proof is analogous to [5, Lemma 8.20]: for such μ and $E \in \mathcal{A}_{\leq 1}$ with $[E] \leq \gamma$ in N_1, such that $E \in \mathcal{T}_{\zeta,(\mu,\eta)}$, we must have $E \in \mathcal{A}_0$. □

7.4 Crossing a wall

Let $\mu \in V_\gamma$. First, we show that we can enter the wall $\{\mu\} \times \mathbb{R}$ from either side in the following sense.

Lemma 7.10. Let $\alpha \in N_{\leq 1}$ and $0 < \varepsilon \ll 1$.

(i) For sufficiently large $\eta \gg 0$

$$\mathrm{Pairs}_\alpha^{\zeta,(\mu,\eta)} = \mathrm{Pairs}_\alpha^{\zeta,(\mu+\varepsilon,\eta)} ,$$

(ii) for sufficiently small $\eta \ll 0$

$$\mathrm{Pairs}_\alpha^{\zeta,(\mu,\eta)} = \mathrm{Pairs}_\alpha^{\zeta,(\mu-\varepsilon,\eta)} .$$

Proof. The proof is a simplified version of [5, Lemma 8.25]. □

We explain now the wall-crossing inside $\{\mu\} \times \mathbb{R}$. Let $c_\mu \in N_0$ be the unique class of Lemma 7.3. For any $c \in N_0$ define

$$^p\mathrm{DT}_{\gamma,c+\mathbb{Z}c_\mu}^{\zeta,(\mu,\eta)} = \sum_{k\in\mathbb{Z}} {}^p\mathrm{DT}_{\gamma,c+kc_\mu}^{\zeta,(\mu,\eta)} z^{c+kc_\mu} \in \mathbb{Q}[[Q^{\pm 1}, q^{\pm 1}]] .$$

We have used the Novikov parameter z to track both q and Q. By the previous lemma, the notion of (μ,η)-pair is constant for $\eta \gg 0$ (respectively $\eta \ll 0$) and fixed $\alpha \in N_{\leq 1}$. Thus, we can choose arbitrary $\eta_0 \in \mathbb{R}$ and define the limit for $\eta \to \pm\infty$, which agrees with the generating series for $(\mu \pm \varepsilon, \eta_0)$:

$$^p\mathrm{DT}_{\gamma,c+\mathbb{Z}c_\mu}^{\zeta,(\mu,\pm\infty)} = {}^p\mathrm{DT}_{\gamma,c+\mathbb{Z}c_\mu}^{\zeta,(\mu\pm\varepsilon,\eta_0)} .$$

Lemma 7.11. The two generating series ${}^p\mathrm{DT}_{\gamma,c+\mathbb{Z}c_\mu}^{\zeta,(\mu,\pm\infty)}$ are the expansion of the same rational function.

Proof. The combinatorics is the same as in Section 6.2, see also [5, Corollary 8.28].

The technical conditions to apply the wall-crossing formula are verified using Proposition 4.19 and Lemma 7.1 in essentially the same way as we did in the proof of Lemma 6.4. For condition (iii) of Section 3.4 we note that if $\sum_{i=1}^n c_i = c$ is fixed and each c_i belongs to a L_μ-bounded set, then there are only finitely many possibilities for each c_i. □

The main result of this section is then a formal consequence.

Proposition 7.12. There exists a rational function $f_\gamma(q,Q)$ such that for all $\mu \in V_\gamma$ the series ${}^p\mathrm{DT}_\gamma^{\zeta,(\mu\pm\varepsilon,\eta)}$ are the expansion of f_γ with respect to $L_{\mu\pm\varepsilon}$.

Proof. Let $\mu = \max V_\gamma$ be the biggest wall and $c_\mu \in N_0$ the class of Lemma 7.3. By Lemma 7.10 and Section 7.3 the series ${}^p\mathrm{DT}_\gamma^{\zeta,(\mu+\varepsilon,\eta)}$ agrees with perverse stable pairs ${}^p\mathrm{PT}_\gamma$ and it is the expansion of a rational function f_γ^μ as proven in Section 6. Note that in the limit $\mu' \to \infty$ the linear function

$$L_{\mu'}(c) = d(c) + \frac{j}{\mu'(a_0)}$$

agrees with $d(-)$ in the sense that expansion of the rational function f_γ^μ is the same for $L_{\mu'}$ and d.

The previous lemma says that the two series ${}^p\mathrm{DT}_{\gamma,c+\mathbb{Z}c_\mu}^{\zeta,(\mu,\pm\infty)}$ agree as rational function for each $c \in N_0$. Their difference is a quasi-polynomial function in k. Recall that, by definition of c_μ, we have

$$L_{\mu+\varepsilon}(c_\mu) > 0\,, \quad L_{\mu-\varepsilon}(c_\mu) < 0\,.$$

It is then a formal consequence [5, Lemma 2.22] that ${}^p\mathrm{DT}_\gamma^{\zeta,(\mu-\varepsilon,\eta)}$ is the expansion of the same rational function f_γ^μ, with respect to $L_{\mu-\varepsilon}$.

Since stability is constant on $\left(\mathbb{R}_{>0} \setminus V_\gamma\right) \times \mathbb{R}$ we can argue by induction on the finite set of walls $\mu' \in V_\gamma$. In particular, we obtain the same rational function f_γ for each wall. □

The limit of ζ-stability for $0 < \mu \ll 1$ was found to agree with BS stability in Section 7.2 which, together with Section 5, concludes the proof of Theorem 1.3.

8 Gromov–Witten theory

In this section we assume the GW/PT correspondence for X. Let

$$R = \mathbb{C}\left[Q^{\pm 1}, \left(\frac{1}{1-Q^j}\right)_{j\geq 1}\right][u^{-1}, u]]$$

and

$$R_a = \left\{f \in R : f(Q,u) = Q^a f(Q^{-1}, -u)\right\}.$$

More explicitly, elements of R are written as

$$f(u,Q) = \sum_{s\geq H} f_s(Q) u^s$$

where $f_s(Q)$ are rational functions of the form

$$f_s(Q) = \frac{p(Q)}{\prod_j (1-Q^{a_j})}$$

with $p(Q)$ a Laurent polynomial. Then $f \in R_a$ if and only if

$$Q^a f_s(Q^{-1}) = (-1)^s f_s(Q)\,.$$

Proposition 8.1. For all $\beta \in N_1(X)$, after the change of variables $q = e^{iu}$ we have

$$^p\mathrm{PT}_\beta(e^{iu}, Q) \in R_{\mathsf{w}\cdot\beta}\,.$$

Proof. We prove first that $^p\mathrm{PT}_\beta \in R$. By Theorem 1.2 it holds that

$$^p\mathrm{PT}_\beta \in \mathbb{Q}\left[q^{\pm 1}, Q^{\pm 1}, \left(\frac{1}{1-q^aQ^b}\right)_{a,b\geq 0}\right].$$

Since clearly $q^{\pm 1}, Q^{\pm 1} \in R$ it suffices to show that $\frac{1}{1-q^aQ^b} \in R$, which follows from the following simple computation:

$$\begin{aligned}\frac{1}{1-e^{iau}Q^b} &= \sum_{k\geq 0} e^{ikau}Q^{kb} = \sum_{k,s\geq 0} u^s \frac{(ia)^s}{s!} k^s Q^{kb}\\ &= \sum_{s\geq 0} u^s \frac{(ia)^s}{s!} \mathrm{Li}_{-s}(Q^b).\end{aligned}$$

Since the polylogarithm $\mathrm{Li}_{-s}(Q)$ is a rational function with denominator $(1-Q)^{s+1}$ for $s \geq 0$, the claim follows.

The rest of the Proposition follows from the functional equation part of Theorem 1.2:

$$Q^{\mathsf{w}\cdot\beta}\, {}^p\mathrm{PT}_\beta(q^{-1}, Q^{-1}) = {}^p\mathrm{PT}_\beta(q, Q)\,.$$

After the change of variables $q = e^{iu}$, it follows that $^p\mathrm{PT}_\beta \in R_{\mathsf{w}\cdot\beta}$. □

Conjecture 8.2. The Proposition above still holds if we replace R by the smaller ring

$$R = \mathbb{C}\left[Q^{\pm 1}, \frac{1}{1-Q}\right][u^{-1}, u]].$$

We now deal with $\mathrm{PT}_0(q, Q)$. This requires that we exclude genus 0 and 1 terms. More precisely, define

$$\widetilde{\mathrm{PT}}_0(q, Q) = \mathrm{PT}_0(q, Q) \cdot \exp\left(\frac{2-2g}{u^2}\mathrm{Li}_3(Q) + \frac{1-g}{6}\mathrm{Li}_1(Q)\right).$$

Proposition 8.3. After the change of variables $q = e^{iu}$, one has

$$\widetilde{\mathrm{PT}}_0(e^{iu}, Q) \in R_0.$$

Proof. We have

$$\mathrm{PT}_0(q,Q) = \exp\left(\sum_{k\geq 1} \frac{(2-2g)(qQ)^k}{k(1-q^k)^2}\right).$$

Writing c_s for the coefficients in the u-expansion of

$$\frac{(2-2g)e^{iu}}{(1-e^{iu})^2} = \sum_{s\geq -2} c_s u^s$$

one has the formula

$$\mathrm{PT}_0(q,Q) = \exp\left(\sum_{s\geq -2} c_s u^s \mathrm{Li}_{1-s}(Q)\right).$$

As easy inspection shows that $c_{-2} = 2g-2$, $c_{-1} = 0 = c_1$, and $c_0 = (g-1)/6$. Thus, the definition of $\widetilde{\mathrm{PT}}_0$ removes the first terms in the previous formula and we find that

$$\widetilde{\mathrm{PT}}_0(q,Q) = \exp\left(\sum_{s\geq 2} c_s u^s \mathrm{Li}_{1-h}(Q)\right).$$

This concludes the proof since, for $s \geq 2$, $\mathrm{Li}_{1-s}(Q)$ is a rational function with denominator $(1-Q)^s$ and satisfies the symmetry property

$$\mathrm{Li}_{1-s}(Q^{-1}) = (-1)^s \mathrm{Li}_{1-s}(Q).$$ □

We provide now the proof of Corollary 1.4. We denote

$$\widetilde{\mathrm{PT}}_\beta(q,Q) = \mathrm{PT}_\beta(q,Q) \cdot \exp\left(\frac{2-2g}{u^2}\mathrm{Li}_3(Q) + \frac{1-g}{6}\mathrm{Li}_1(Q)\right).$$

By Theorem 1.3,

$${}^{p}\mathrm{PT}_\beta(q,Q)\, \widetilde{\mathrm{PT}}_0(q,Q) = \widetilde{\mathrm{PT}}_\beta(q,Q)\,,$$

so Propositions 8.1 and 8.3 together imply that $\widetilde{\mathrm{PT}}_\beta(q,Q) \in R_{\mathsf{w}\cdot\beta}$. Hence the generating function

$$\sum_{\beta\in N_1(X)} \widetilde{\mathrm{PT}}_\beta(q,Q)\, z^\beta$$

belongs to the ring

$$\mathcal{R} = \left\{ \sum_{\beta\in N_1(X)} f_\beta(q,Q)\, z^\beta : f_\beta \in R_{\mathsf{w}\cdot\beta} \right\}.$$

Moreover, with the usual change of variable $q = e^{iu}$, we have

$$\exp\left(\sum_{(h,\beta)\neq(0,0),(1,0)} u^{2h-2} z^{\beta} \sum_{j\in\mathbb{Z}} \mathrm{GW}_{h,\beta+j\mathsf{b}}\, Q^j\right) = \sum_{\beta\in N_1(X)} \widetilde{\mathrm{PT}}_{\beta}(q,Q)\, z^{\beta} \in \mathcal{R}.$$

Taking the logarithm preserves $\mathcal{R}$, finishing the proof of Corollary 1.4.

Local Hirzebruch surface

In this appendix we take a closer look at the non-compact Calabi–Yau 3-fold K_W associated to the Hirzebruch surface $W = \mathbb{F}_r$. We use the topological vertex to compute their enumerative invariants. In particular, we prove the following strengthening of Corollary 1.4 in the local case:

Theorem 8.4. Let $X = K_W$ be a local Hirzebruch surface. For all $h \in \mathbb{Z}_{\geq 0}$ and $\beta \in H_2(W,\mathbb{Z})$ such that $(h,\beta) \neq (0, m\mathsf{b})\,, (1, m\mathsf{b})$, the series

$$\sum_{j\in\mathbb{Z}} \mathrm{GW}^X_{h,\beta+j\mathsf{b}}\, Q^j$$

is the expansion of a rational function $f_{h,\beta}(Q)$ of the form

$$f_{h,\beta}(Q) = \frac{p_{h,\beta}(Q)}{(1-Q)^{4(\mathsf{b}\cdot\beta)+2h-2}}\,,$$

where $p_{h,\beta}$ is a Laurent polynomial. Moreover, $f_{h,\beta}$ satisfies the functional equation

$$f_{h,\beta}(Q^{-1}) = Q^{-K_W\cdot\beta}\, f_{h,\beta}(Q)\,.$$

In the theorem, the intersection products $\mathsf{b}\cdot\beta$ and $K_W\cdot\beta$ are taken in $H^*(W)$. The canonical class is

$$K_W = -2\mathsf{c} - (2+r)\mathsf{b}\,,$$

where c is the class of the torus-invariant section with non-positive self-intersection $\mathsf{c}^2 = -r$.

Remark 8.5. The form of the rational function implies that if we fix k, h, r then $\mathrm{GW}^{K_{\mathbb{F}_r}}_{h,m\mathsf{c}+j\mathsf{b}}$ is a polynomial in j of degree $4m + 2h - 3$ for large enough j. In [23, Equation 5.2] the authors predict the assymptotic behavior for $h = 0$:

$$\mathrm{GW}^{K_{\mathbb{F}_r}}_{h=0,m\mathsf{c}+j\mathsf{b}} \sim \gamma_m j^{4m-3}$$

for some constant γ_m that doesn't depend on r. The independence of r is not difficult to see from our proof.

8.1 Combinatorics of the 2-leg topological vertex

The local Hirzebruch surface K_W is a toric non-compact Calabi–Yau 3-fold, so its Pandharipande–Thomas invariants can be computed via the formalism of the topological vertex. The 2-leg case of the topological vertex admits simple combinatorical expressions, also known as Iqbal's formula [21, 30, 52, 54]. We now describe such formula.

Given a partition μ, we associate to it the Schur function $s_\mu(x_1, \dots, x_n)$ (see for example [31, I.3]). An explicit way to define s_μ is the following:

$$s_\mu = \det\left(h_{\mu_i - i + j}\right)_{1 \le i,j \le N}$$

where $N \ge \ell(\mu)$ and $h_k = h_k(x_1, \dots, x_n)$ are the complete homogeneous polynomials. We will often consider the specialization of s_μ to the infinite set of variables $x = (1, q, q^2, \dots)$. In this case the definition above is still valid with

$$h_k(1, q, q^2, \dots) = \prod_{j=1}^{k} \frac{1}{1 - q^j}$$

for $k \ge 0$ and $h_k = 0$ for $k < 0$. An alternative way to write $s_\mu(1, q, q^2, \dots)$ is the hook-content product formula

$$s_\mu(1, q, q^2, \dots) = q^{n(\mu)} \prod_{\square \in \mu} \frac{1}{1 - q^{h(\square)}} .$$

In the formula, $n(\mu) = \sum_{i=1}^{\ell(\mu)} (i-1)\mu_i$. Th product runs over boxes in the Young diagram of μ and $h(\square)$ is the hook length.

Iqbal introduced $\mathcal{W}$-functions that play a role in the 1-leg and 2-leg vertex formulas. For a partition μ, it is defined as

$$\mathcal{W}_\mu(q) = (-1)^{|\mu|} q^{k(\mu)/2 + |\mu|/2}\, s_\mu(1, q, q^2, \dots) ,$$

where

$$k(\mu) = \sum_{i=1}^{\ell(\mu)} \mu_i(\mu_i - 2i + 1) \in \mathbb{Z} .$$

For two partitions μ, ν we define

$$\mathcal{W}_{\mu,\nu}(q) = q^{|\nu|/2}\, \mathcal{W}_\mu(q)\, s_\nu(q^{\mu_1 - 1}, q^{\mu_2 - 2}, \dots) .$$

Although it is not apparent from this definition, we have symmetry in the two partitions, i.e. $\mathcal{W}_{\mu,\nu} = \mathcal{W}_{\nu,\mu}$ [53, Theorem 5.1].

We can now formulate Iqbal's formula for the Gromov-Witten invariants of local toric surfaces. Let W be a toric surface. Let

$$D_1\,, D_2\,, \dots\,, D_N\,, D_{N+1} = D_1$$

be the toric divisors in the order they appear in the moment polygon of W. Denote $s_j = D_j^2 \in \mathbb{Z}$ the self-intersection numbers.

Theorem 8.6 ([52, Theorem 1])**.** The partition function for the disconnected Gromov–Witten invariants of K_W is

$$Z^{K_W} = \sum_{\mu_1,\dots,\mu_N} \prod_{j=1}^{N} \Big((-1)^{s_j|\mu_j|}\, q^{k(\mu_j)s_j/2}\, \mathcal{W}_{\mu_j,\mu_{j+1}}(q)\, z^{|\mu_j| D_j} \Big)$$

after the change of variables $q = e^{iu}$.

Recall that under the change of variables $q = e^{iu}$ we have

$$Z^{K_W} = \mathrm{PT}^{K_W}(q,z) = \sum_{n,\beta} \mathrm{PT}_{\beta,n}\, z^{\beta}\, (-q)^n\,.$$

8.2 Iqbal's formula for Hirzebruch surfaces

We specialize Theorem 8.6 to the case of the Hirzebruch surface $W = \mathbb{F}_r$. The homology $H_2(W,\mathbb{Z})$ is generated by two classes b and c where b is the fiber class and c is the class of the torus-invariant section $\mathbb{P}^1 \hookrightarrow W$ with non-positive self-intersection $\mathsf{c}^2 = -r$. The toric divisors of W are

$$D_1 = \mathsf{b} = D_3\,, \quad D_2 = \mathsf{c} + r\mathsf{b}\,, \quad D_4 = \mathsf{c}\,.$$

We denote by $Q = z^{\mathsf{b}}$ and $Q_{\mathsf{c}} = z^{\mathsf{c}}$ the Novikov variables, then

$$\begin{aligned} Z^{K_W} &= \sum_{\mu_1,\dots,\mu_4} \big(q^{r(k(\mu_2)-k(\mu_4))} \mathcal{W}_{\mu_1,\mu_2} \mathcal{W}_{\mu_2,\mu_3} \mathcal{W}_{\mu_3,\mu_4} \mathcal{W}_{\mu_4,\mu_1} \\ &\qquad \times ((-1)^r Q_{\mathsf{c}})^{|\mu_2|+|\mu_4|} Q^{|\mu_1|+|\mu_3|+r|\mu_2|} \big) \\ &= \sum_{m=0}^{\infty} Q_{\mathsf{c}}^{j} (-1)^{rm} \sum_{|\mu_2|+|\mu_4|=m} \Bigg(q^{r(k(\mu_2)-k(\mu_4))} Q^{r|\mu_2|} \\ &\qquad \times \Big(\sum_{\lambda} \mathcal{W}_{\mu_2,\lambda} \mathcal{W}_{\mu_4,\lambda} Q^{|\lambda|} \Big)^2 \Bigg) \end{aligned} \tag{2.8}$$

The sum appearing in the last line

$$S_{\mu,\nu}(q,Q) = \sum_{\lambda} \mathcal{W}_{\mu\lambda}(q)\mathcal{W}_{\nu\lambda}(q)Q^{|\lambda|} \in \mathbb{Q}((q,Q))$$

admits a nice closed formula [15, Proposition 1]. We give a proof which is a bit more direct than the one in [15]. Let

$$p_\mu(q) = \sum_{i=1}^{\infty} q^{\mu_i - i} = \frac{q^{-\ell(\mu)}}{q-1} + \sum_{i=1}^{\ell(\mu)} q^{\mu_i - i}\,.$$

Lemma 8.7 ([15, Proposition 1])**.** For any two partitions μ, ν we have the following identity in $\mathbb{Q}((q,Q))$:

$$S_{\mu,\nu} = \mathcal{W}_\mu \mathcal{W}_\nu \exp\left(\sum_{k=1}^{\infty} p_\mu(q^k)p_\nu(q^k)\frac{(qQ)^k}{k}\right)\,. \tag{2.9}$$

Proof. Let $x_i = (qQ)^{1/2}q^{\mu_i - i}$, $y_j = (qQ)^{1/2}q^{\nu_j - j}$. Then

$$p_\mu(q^k) = (qQ)^{-k/2}\sum_{i\geq 1} x_i^k = (qQ_F)^{-k/2}P_k(x)\,,$$

where $P_k(x)$ is the k-th power function. For a partition λ let

$$P_\lambda(x) = \prod P_{\lambda_i}(x)\,, \quad m_k = \#\{i : \lambda_i = k\}\,, \quad z_\lambda = \prod k^{m_k} m_k!\,.$$

By expanding the exponential and cancelling $\mathcal{W}_\mu\mathcal{W}_\nu$ on both sides, using

$$\mathcal{W}_{\mu,\lambda} = q^{|\lambda|/2}\,\mathcal{W}_\mu\, s_\lambda(q^{\mu_1 - 1}, q^{\mu_2 - 2}, \ldots)\,,$$

we're left to show

$$\sum_\lambda (qQ)^{|\lambda|} s_\lambda(q^{\mu_1-1}, q^{\mu_2-2}, \ldots)s_\lambda(q^{\nu_1-1}, q^{\nu_2-2}, \ldots)$$
$$= \sum_\lambda \prod_{k=1}^{\ell(\lambda)} \frac{1}{m_k!}\left(\frac{P_k(x)P_k(y)}{k}\right)^{m_k}.$$

By the Cauchy identity [31, Eq. 4.3] the LHS is

$$\prod_{i,j\geq 1}\frac{1}{1-x_iy_j}\,,$$

and the RHS is

$$\sum_\lambda z_\lambda^{-1}P_\lambda(x)P_\lambda(y)\,.$$

The two sides agree [31, Eq. 4.1, 4.3]. □

8.3 Rationality of $\mathrm{PT}_\beta/\mathrm{PT}_0$

We now give a quick proof of the rationality result in Theorem 1.1 in the local case based on our computations. Equation (2.9) can also be written as an infinite product formula in the following way. We can write

$$p_\mu(q)p_\nu(q) = \frac{1}{(1-q)^2}\sum_{i=-s}^{s} a_i\, q^i$$

for some $s, a_i \in \mathbb{Z}_{\geq 0}$ depending on μ, η. Then,

$$S_{\mu,\nu} = \mathcal{W}_\mu \mathcal{W}_\nu \prod_{i=-s}^{s}\left(\prod_{j\geq 1}(1-q^{j+i}Q)^{-j}\right)^{a_i}.$$

Note in particular that taking the constant Q_{c}^0 coefficient in equation (2.8) we find

$$\mathrm{PT}_0(q,Q) = [Q_{\mathsf{c}}^0]Z^{K_W} = S_{\emptyset\emptyset}^2 = \prod_{j\geq 1}(1-q^jQ)^{-2j}.$$

Since $\mathcal{W}_\mu, \mathcal{W}_\nu \in \mathbb{Q}(q)$ and

$$\sum_{i=-s}^{s} a_i = 1\,, \quad \sum_{i=-s}^{s} i\, a_i = 0\,,$$

one can see that

$$\frac{S_{\mu\nu}}{S_{\emptyset\emptyset}} \in \mathbb{Q}(q,Q)\,.$$

Together with equation (2.8) it follows that

$$\frac{\mathrm{PT}_{m\mathsf{c}}(q,Q)}{\mathrm{PT}_0(q,Q)} = [Q_{\mathsf{c}}^m]\frac{Z^{K_W}}{S_{\emptyset\emptyset}^2} \in \mathbb{Q}(q,Q)\,.$$

8.4 Proof of Theorem 8.4

We give the proof of Theorem 8.4 based on the application of Iqbal's formula (2.8). We first remark that it is enough to prove the result when $\beta = m\mathsf{c}$ for some $m \geq 0$. Indeed, if $\widetilde{\beta} = \beta + k\mathsf{b}$ then the corresponding generating functions are related by multiplication by Q^{-k} and

$$\mathsf{b}\cdot\widetilde{\beta} = \mathsf{b}\cdot\beta\,, \quad -K_W\cdot\widetilde{\beta} = -K_W\cdot\beta + 2k\,.$$

We define a refinement $R_{a,b} \subset R_a$ of the sets introduced in Section 8. Elements of $R_{a,b}$ are Laurent series of the form

$$f(Q,u) = \sum_{s \geq H} f_s(Q) u^s$$

such that $f_s(Q)$ take the form

$$f_s(Q) = \frac{p_s(Q)}{(1-Q)^{b+s}}$$

and satisfy

$$Q^a f_s(Q^{-1}) = (-1)^s f_s(Q).$$

For a Laurent series $f(q,Q)$ in variables q, Q we say that $f \in R_{a,b}$ if it is in $R_{a,b}$ after the change of variables $q = e^{iu}$. We must show that

$$\widetilde{\mathrm{PT}}_{m\mathsf{c}}(q,Q) \in R_{2m,(2+r)m}\,,$$

where $\widetilde{\mathrm{PT}}$ is as defined in Section 8. We consider the u-expansion of the series

$$p_\mu(e^{iu}) p_\nu(e^{iu}) e^{iu} = \sum_{s=-2}^{\infty} c_s^{\mu\nu} u^n\,.$$

The first few terms of the expansion are easily computed:

$$p_\mu(e^{iu}) p_\nu(e^{iu}) e^{iu} = -u^{-2} + \left(|\mu| + |\nu| - \frac{1}{12}\right) + \frac{iu}{2}\big(k(\mu) + k(\nu)\big) + O(u^2)\,.$$

Plugging the expansion into equation (2.9) we get

$$S_{\mu,\nu} = \mathcal{W}_\mu \mathcal{W}_\nu \exp\left(\sum_{s=-2}^{\infty} c_s^{\mu\nu} u^s \mathrm{Li}_{1-s}(Q)\right)\,.$$

Defining now the modification

$$\widetilde{S}_{\mu,\nu} = S_{\mu,\nu} \exp\left(\frac{1}{u^2}\mathrm{Li}_3(Q) + \frac{1}{12}\mathrm{Li}_1(Q)\right)$$

we have the formula

$$\widetilde{S}_{\mu,\nu} = (1-Q)^{-(|\mu|+|\nu|)} \widetilde{\mathcal{W}}_\mu \widetilde{\mathcal{W}}_\nu \exp\left(\frac{iu}{4}\big(k(\mu)+k(\nu)\big)\frac{1+Q}{1-Q}\right)$$
$$\times \exp\left(\sum_{s=2}^{\infty} c_s^{\mu\nu} u^s \mathrm{Li}_{1-s}(Q)\right)\,.$$

where

$$\widetilde{\mathcal{W}}_\mu(q) = q^{-\frac{k(\mu)}{4}} \mathcal{W}_\mu(q) = \exp\left(\frac{iu}{4} k(\mu)\right) \mathcal{W}_\mu(q)\,.$$

We used the identities

$$\mathrm{Li}_1(Q) = -\log(1-Q)\,, \quad \mathrm{Li}_0(Q) = \frac{Q}{1-Q}\,.$$

For $s \geq 2$, $\mathrm{Li}_{1-s}(Q)$ is a rational function with denominator $(1-Q)^s$ and satisfies the symmetry property

$$\mathrm{Li}_{1-s}(Q^{-1}) = (-1)^s \mathrm{Li}_{1-s}(Q)\,.$$

Moreover, $\widetilde{\mathcal{W}}$ satisfies $\widetilde{\mathcal{W}}(q) = \widetilde{\mathcal{W}}(1/q)$ (see [53, Proposition 5.1]) so we have, for $m = |\mu| + |\nu|$,

$$\widetilde{S}_{\mu,\nu} \in R_{m,m}\,.$$

We can now finish the proof of Theorem 8.4. From equation (2.8) we have

$$\widetilde{\mathrm{PT}}_{m\mathsf{c}}(q,Q) = (-1)^{rm} \sum_{|\mu_2|+|\mu_4|=m} \left(q^{r(k(\mu_2)-k(\mu_4))} Q^{r|\mu_2|} \widetilde{S}^2_{\mu_2,\mu_4} \right).$$

We pair the (μ_2, μ_4) and (μ_4, μ_2) terms and note that

$$q^{r(k(\mu_2)-k(\mu_4))} Q^{r|\mu_2|} + q^{r(k(\mu_4)-k(\mu_2))} Q^{r|\mu_4|} \in R_{0,rm}\,,$$

therefore

$$\widetilde{\mathrm{PT}}_{m\mathsf{c}}(q,Q) \in R_{2m,(2+r)m}\,.$$

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Chapter 3

Curves on K3 surfaces in divisibility 2

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1 Introduction

Let S be a complex nonsingular projective K3 surface and $\beta \in H_2(S,\mathbb{Z})$ an effective curve class. Gromov–Witten invariants of S are defined via intersection theory on the moduli space $\overline{M}_{g,n}(S,\beta)$ of stable maps from n-pointed genus g curves to S. This moduli space comes with a virtual fundamental class. However, the virtual class vanishes for $\beta \neq 0$ so, instead, we use the *reduced class*[1]

$$[\overline{M}_{g,n}(S,\beta)]^{red} \in A_{g+n}\big(\overline{M}_{g,n}(S,\beta),\mathbb{Q}\big)\,.$$

For integers $a_i \geq 0$ and cohomology classes $\gamma_i \in H^*(S,\mathbb{Q})$ we define

$$\big\langle \tau_{a_1}(\gamma_1)\dots\tau_{a_n}(\gamma_n)\big\rangle^S_{g,\beta} = \int_{[\overline{M}_{g,n}(S,\beta)]^{red}} \prod_{i=1}^n \psi_i^{a_i} \cup \mathrm{ev}_i^*(\gamma_i)\,,$$

where $\mathrm{ev}_i\colon \overline{M}_{g,n}(S,\beta) \to S$ is the evaluation at i-th marking and ψ_i is the cotangent class at the i-th marking. By the deformation invariance of the reduced class, the invariant only depends on the norm $\langle \beta,\beta\rangle$ and the divisibility of the curve class β.

1.1 Quasimodularity

Gromov–Witten invariants of K3 surfaces for primitive curve classes are well-understood since the seminal paper by Maulik, Pandharipande, and Thomas [29]. The invariants are coefficients of weakly holomorphic[2] quasimodular forms with pole of order at most one [29, Theorem 4]. For imprimitive curve classes, the quasimodularity is conjectured with the level structure [29, Section 7.5].

[1]We will identify this class with its image under the cycle class map $A_* \to H_{2*}$.

[2]Weakly holomorphic means holomorphic on the upper half plane with possible pole at the cusp $i\infty$.

The quasimodularity can be stated in a precise sense via elliptic K3 surfaces. Let

$$\pi\colon S \to \mathbb{P}^1$$

be an elliptic K3 surface with a section and denote by $B, F \in H_2(S, \mathbb{Z})$ the class of the section resp. a fiber. For any $m \geq 1$ one defines the descendent potential

$$\mathsf{F}_{g,m}\big(\tau_{a_1}(\gamma_1)\dots\tau_{a_n}(\gamma_n)\big) = \sum_{h\geq 0} \big\langle \tau_{a_1}(\gamma_1)\dots\tau_{a_n}(\gamma_n)\big\rangle^S_{g,\,mB+hF}\, q^{h-m}\,.$$

Note that this generating series involves curve classes $mB + hF$ of different divisibilities, bounded by m.

It is convenient to use the following homogenized insertions which will lead to quasimodular forms of pure weight. Let $\mathsf{1} \in H^0(S)$ and $\mathsf{p} \in H^4(S)$ be the identity resp. the point class. Denote

$$W = B + F \in H^2(S)$$

and let

$$U = \mathbb{Q}\langle F, W\rangle \subset H^2(S)$$

be the hyperbolic plane in $H^2(S)$ and let $U^\perp \subset H^2(S)$ be its orthogonal complement with respect to the intersection form. We only consider second cohomology classes which are pure with respect to the decomposition

$$H^2(S, \mathbb{Q}) \cong \mathbb{Q}\langle F\rangle \oplus \mathbb{Q}\langle W\rangle \oplus U^\perp\,.$$

Following [8, Section 4.6], define a modified degree function $\underline{\deg}$ by

$$\underline{\deg}(\gamma) = \begin{cases} 2 & \text{if } \gamma = W \text{ or } \mathsf{p}\,, \\ 1 & \text{if } \gamma \in U^\perp\,, \\ 0 & \text{if } \gamma = F \text{ or } \mathsf{1}\,. \end{cases}$$

For $m \geq 1$, consider the *Hecke congruence subgroup of level m*

$$\Gamma_0(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \mod m \right\}$$

and let $\mathsf{QMod}(m)$ be the space of quasimodular forms for the congruence subgroup $\Gamma_0(m) \subset \mathrm{SL}_2(\mathbb{Z})$. Let $\Delta(q)$ be the modular discriminant

$$\Delta(q) = q\prod_{n\geq 1}(1-q^n)^{24}\,.$$

Our first main result proves level two quasimodularity of $\mathsf{F}_{g,2}$, previously conjectured by Maulik, Pandharipande, and Thomas [29, Section 7.5].

Theorem 1.1. Let $\gamma_1, \ldots, \gamma_n \in H^*(S)$ be homogeneous on the modified degree function $\underline{\deg}$. Then $\mathsf{F}_{g,2}$ is the Fourier expansion of a quasimodular form

$$\mathsf{F}_{g,2}\big(\tau_{a_1}(\gamma_1)\ldots\tau_{a_n}(\gamma_n)\big) \in \frac{1}{\Delta(q)^2}\mathsf{QMod}(2)$$

of weight $2g - 12 + \sum_i \underline{\deg}(\gamma_i)$ with pole at $q = 0$ of order at most 2.

1.2 Holomorphic anomaly equation

In the physics literature, the (conjectural) *holomorphic anomaly equation* [4, 5] predicts hidden structures of the Gromov–Witten partition function associated to Calabi–Yau varieties. For the past few years, there has been an extensive work to prove the holomorphic anomaly equation in many cases: local $\mathbb{P}^2$ [26], the quintic threefold [11, 16], K3 surface with primitive curve classes [33], elliptic fibration [34] and $\mathbb{P}^2$ relative to a smooth cubic [6].

Every quasimodular form for $\Gamma_0(m)$ can be written uniquely as a polynomial in C_2 with coefficients which are modular forms for $\Gamma_0(m)$ [18, Proposition 1]. Here,

$$C_2(q) = -\frac{1}{24}E_2(q)$$

is the renormalized second Eisenstein series. Assuming quasimodularity, the holomorphic anomaly equation fixes the non-holomorphic parameter of the Gromov–Witten partition function of K3 surfaces in terms of lower weight partition functions: it computes the derivative of $\mathsf{F}_{g,m}$ with respect to the C_2 variable. See [33] for the proof of holomorphic anomaly equation for K3 surfaces with primitive curve classes and [34] for the holomorphic anomaly equation associated to elliptic fibrations.

Define an endomorphism [33, Section 0.6]

$$\sigma\colon H^*(S^2) \to H^*(S^2)$$

by the following assignments:

$$\sigma(\gamma \boxtimes \gamma') = 0$$

if γ or $\gamma' \in H^0(S) \oplus \mathbb{Q}\big\langle F\big\rangle \oplus H^4(S)$, and for $\alpha, \alpha' \in U^\perp$,

$$\sigma(W \boxtimes W) = \Delta_{U^\perp},\ \sigma(W \boxtimes \alpha) = -\alpha \boxtimes F,$$
$$\sigma(\alpha \boxtimes W) = -F \boxtimes \alpha,\ \sigma(\alpha, \alpha') = \langle \alpha, \alpha'\rangle F \boxtimes F\,,$$

where $\Delta_{U^\perp}$ denotes the diagonal class for the intersection pairing on $U^\perp$. We will view σ as the exterior product $\sigma_1 \boxtimes \sigma_2$ via Künneth decomposition.

Recall the virtual fundamental class for trivial curve classes which will play a role for the holomorphic anomaly equation. For $\beta = 0$ we have an isomorphism

$$\overline{M}_{g,n}(S,0) \cong \overline{M}_{g,n} \times S$$

and the virtual class is given by

$$[\overline{M}_{g,n}(S,0)]^{vir} = \begin{cases} [\overline{M}_{0,n} \times S] & \text{if } g = 0\,, \\ c_2(S) \cap [\overline{M}_{1,n} \times S] & \text{if } g = 1\,, \\ 0 & \text{if } g \geq 2\,. \end{cases}$$

Also, consider the pullback under the morphism $\pi\colon S \to \mathbb{P}^1$ of the diagonal class of $\mathbb{P}^1$

$$\Delta_{\mathbb{P}^1} = 1 \boxtimes F + F \boxtimes 1 = \sum_{i=1}^{2} \delta_i \boxtimes \delta_i^\vee\,.$$

Define the generating series[3]

$$\begin{aligned} &\mathsf{H}_{g,m}\big(\alpha;\gamma_1,\ldots,\gamma_n\big) \qquad (3.1)\\ &= \mathsf{F}_{g-1,m}\big(\alpha;\gamma_1,\ldots,\gamma_n,\Delta_{\mathbb{P}^1}\big)\\ &\quad + 2 \sum_{\substack{g=g_1+g_2\\ \{1,\ldots,n\}=I_1\sqcup I_2\\ i\in\{1,2\}}} \mathsf{F}_{g_1,m}\big(\alpha_{I_1};\gamma_{I_1},\delta_i\big)\,\mathsf{F}^{vir}_{g_2}\big(\alpha_{I_2};\gamma_{I_2},\delta_i^\vee\big)\\ &\quad - 2\sum_{i=1}^{n} \mathsf{F}_{g,m}\big(\alpha\psi_i;\gamma_1,\ldots,\gamma_{i-1},\pi^*\pi_*\gamma_i,\gamma_{i+1},\ldots,\gamma_n\big)\\ &\quad + \frac{20}{m}\sum_{i=1}^{n}\langle\gamma_i,F\rangle \mathsf{F}_{g,m}\big(\alpha;\gamma_1,\ldots,\gamma_{i-1},F,\gamma_{i+1},\ldots,\gamma_n\big)\\ &\quad - \frac{2}{m}\sum_{i<j} \mathsf{F}_{g,m}\big(\alpha;\gamma_1,\ldots,\underbrace{\sigma_1(\gamma_i,\gamma_j)}_{i\text{th}},\ldots,\underbrace{\sigma_2(\gamma_i,\gamma_j)}_{j\text{th}},\ldots,\gamma_n\big)\,, \end{aligned}$$

where F^{vir} denotes the generating series for virtual fundamental class. In most cases this term vanishes. The equation takes almost the same form for arbitrary m, only the last two terms acquire a factor of $\frac{1}{m}$. The appearance of these factors is explained in Section 4, see also Example 4.2. We conjecture that the holomorphic anomaly equation has the following form:

[3] Here, instead of descendent insertions we use a tautological class $\alpha \in R^*(\overline{M}_{g,n})$, see the comment in Section 3.2

Conjecture 1.2.

$$\frac{d}{dC_2}\mathsf{F}_{g,m}(\alpha;\gamma_1,\ldots,\gamma_n) = \mathsf{H}_{g,m}(\alpha;\gamma_1,\ldots,\gamma_n)\,. \tag{3.2}$$

For primitive curve classes, the holomorphic anomaly equation is proven in [33]. In higher divisiblity, it is precisely equation (3.2) that would be implied by the conjectural multiple cover formula for imprimitve Gromow–Witten invariants of K3 surfaces. We explain this in the following section. We prove Conjecture 1.2 unconditionally when $m = 2$:

Theorem 1.3. For any $g \geq 0$,

$$\frac{d}{dC_2}\mathsf{F}_{g,2}(\alpha;\gamma_1,\ldots,\gamma_n) = \mathsf{H}_{g,2}(\alpha;\gamma_1,\ldots,\gamma_n)\,. \tag{3.3}$$

1.3 Multiple cover formula

Motivated by the Katz–Klemm–Vafa (KKV) formula, Oberdieck and Pandharipande conjectured a formula which computes imprimitive invariants from the primitive invariants:

Conjecture 1.4. ([32, Conjecture C2]) For a primitive curve class β,

$$\begin{aligned}&\big\langle \tau_{a_1}(\gamma_1)\ldots\tau_{a_n}(\gamma_n)\big\rangle_{g,\,m\beta} \\ &= \sum_{d|m} d^{2g-3+\deg}\big\langle \tau_{a_1}(\varphi_{d,m}(\gamma_1))\ldots\tau_{a_n}(\varphi_{d,m}(\gamma_n))\big\rangle_{g,\,\varphi_{d,m}\left(\frac{m}{d}\beta\right)}\,.\end{aligned} \tag{3.4}$$

The invariants on the right hand side are with respect to primitive curve classes[4]. Assuming this formula, we can deduce the holomorphic anomaly equation:

Proposition 1.5. Let $m \geq 1$. Assume the multiple cover formula (3.4) holds for all curve classes of divisibility $d \mid m$ and all descendent insertions. Then the holomorphic anomaly equation (3.2) holds.

Given this proposition, it seems a natural strategy to prove the multiple cover formula in divisibility two and deduce, as a consequence, the holomorphic anomaly equation. Indeed, our method does follow this logic for $m = 2$ and for low genus: we verify the multiple cover formula for $g \leq 2$, see Example 8.2. For higher genus, however, our method does not seem suitable to achieve this. Instead, our proof of Theorem 1.1 provides an algorithm, based on the degeneration to the normal

[4] Section 3 contains all relevant definitions.

cone of a smooth elliptic fiber $E \subset S$, to reduce divisibility two invariants to low genus invariants for which the multiple cover formula is known[5]. The degeneration formula intertwines invariants of S with invariants of $\mathbb{P}^1 \times E$ in a non-trivial way. This phenomenon is illustrated in Example 8.2 for the genus 2 invariants

$$\left\langle \tau_0(\mathsf{p})^2 \right\rangle_{2,\,2\beta} .$$

1.4 Hecke operator

In Section 3 we apply Conjecture 1.4 to an elliptic K3 surface to deduce a conjectural multiple cover formula for the descendent potentials $\mathsf{F}_{g,m}$. The multiple cover formula for any divisibility m is then simply a *Hecke operator of the wrong weight* acting on the primitive potential $\mathsf{F}_{g,1}$. Indeed, the weight of $\mathsf{F}_{g,1}$ (and conjecturally of $\mathsf{F}_{g,m}$) is $2g - 12 + \underline{\deg}$, whereas the Hecke operator has the weight of a descendent potential attached to elliptic curves, namely $2g - 2 + \underline{\deg}$. This operator can be expressed in terms of Hecke operators (of the correct weight) and translation $q \mapsto q^d$. Together with the holomorphic anomaly equation for primitive curve classes [33] this naturally leads to the above conjecture for the holomorphic anomaly equation for higher divisibility.

1.5 Plan of the paper

We prove the quasimodularity and the holomorphic anomaly equation by induction on the genus and the number of markings. In Section 2, we discuss Hecke theory for weakly holomorphic quasimodular forms. This leads to a natural formulation of the multiple cover formula in Section 3 and the imprimitive holomorphic anomaly equation in Section 4. In Section 5, compatibility of the holomorphic anomaly equation with the degeneration formula is presented. In Section 6, we derive the multiple cover formula, which implies the holomorphic anomaly equation, for genus 0, genus 1 and some genus 2 decendent invariants from the KKV formula. The genus 2 computation relies on double ramification relations with target variety. This result serves as the initial condition for our induction. In Section 7, we use previous results to prove Theorem 1.1 and 1.3. The property of the top tautological group $R^{g-1}(M_{g,n})$ reduces higher genus cases to lower genus invariants discussed in Section 6.

[5]The genus 0 and genus 1 cases are proved by Lee and Leung in [24, 25]. Their proof involves a degeneration formula in symplectic geometry which is not possible in algebraic geometry. We present an algebro-geometric approach using the KKV formula.

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2 Quasimodular forms and Hecke operators

We recall basic properties of quasimodular forms and Hecke operators, see [22, 39], in particular [22, pp. 156–163] and [22, Ch. 3, Section 3]. The Hecke theory for weakly holomorphic quasimodular forms however seems to be less well documented. We thus also include some proofs.

The following operators will play a central role. For any Laurent series

$$f(q) = \sum_{n=-\infty}^{\infty} a_n q^n \tag{3.5}$$

and $d \in \mathbb{Z}_{>0}$ we define

$$\mathsf{D}_q f = q\frac{d}{dq} f\,, \quad \mathsf{B}_d f = \sum_{n=-\infty}^{\infty} a_n q^{dn}\,, \quad \mathsf{U}_d f = \sum_{n=-\infty}^{\infty} a_{dn} q^n\,.$$

We will apply these operators to the Laurent series associated to certain modular functions. For this we briefly review the definition of modular forms.

2.1 Quasimodular forms

Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ be the upper half-plane. The group $\mathrm{GL}_2^+(\mathbb{R})$ of real 2×2-matrices with positive determinant acts on $\mathbb{H}$ via

$$A\tau = \frac{a\tau + b}{c\tau + d}\,, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})\,.$$

Let $f\colon \mathbb{H} \to \mathbb{C}$ be a function and let

$$q = e^{2\pi i\tau}\,, \quad y = \mathrm{Im}(\tau)\,.$$

For $k \in \mathbb{Z}$ define the *k-th slash operator*

$$(f|_k A)(\tau) = \det(A)^{k/2}(c\tau + d)^{-k} f(A\tau)\,.$$

Definition 2.1. A *quasimodular form* of weight k for $\mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function $f\colon \mathbb{H} \to \mathbb{C}$ admitting a Fourier expansion

$$f(q) = \sum_{n=0}^{\infty} a_n q^n\,, \quad |q| < 1\,, \tag{3.6}$$

such that there exist $p \geq 0$ and holomorphic functions f_r, $r = 0, \ldots, p$ satisfying the following conditions:

(i) the (non-holomorphic) function $\widehat{f} = \sum_{r=0}^{p} f_r y^{-r}$ satisfies the transformation law

$$\widehat{f}|_k\gamma = \widehat{f} \text{ for all } \gamma \in \mathrm{SL}_2(\mathbb{Z})\,,$$

(ii) $f = f_0$,

(iii) each f_r has an expansion of the form (3.6).

If $p = 0$ then f is called a *modular form*. We denote the space of modular resp. quasimodular forms by Mod and QMod.

Remark 2.2. If $\widehat{f} = \sum_{r=0}^{p} f_r y^{-r}$ as above with $f_p \neq 0$, then each f_r is a quasimodular form of weight $k - 2r$, see [39, Proposition 20]. Moreover, the last one, i.e. f_p is in fact modular (of weight $k - 2p$). The following structural results are well-known [39, Proposition 4, Proposition 20]

$$\mathsf{Mod} = \mathbb{C}[C_4, C_6]\,, \quad \mathsf{QMod} = \mathbb{C}[C_2, C_4, C_6]\,,$$

where

$$C_{2i}(q) = -\frac{B_{2i}}{2i \cdot (2i)!} E_{2i}(q)$$

is the renormalized $2i$-th Eisenstein series. The notion (i) defines the space AHM of *almost holomorphic modular forms* and the assignment $\widehat{f} \mapsto f$ is an isomorphism

$$\mathsf{AHM} \to \mathsf{QMod}\,.$$

Under this map, differentiation with respect to $\frac{1}{8\pi y}$ corresponds to differentiation with respect to C_2.

The modular functions considered in this paper will usually have poles at the cusp $\tau = i\infty$ corresponding to $q = 0$. We will refer to these functions as *weakly holomorphic* with pole of specified order. We want to clarify this terminology in the context of quasimodular forms.

Definition 2.3. A function f is said to be *weakly holomorphic quasimodular with pole of order at most* $m \geq 0$, if f satisfies the conditions in Definition 2.1 except that each f_r is allowed to have a pole at the cusp $i\infty$ of order at most m. If $p = 0$ then f is called a weakly holomorphic modular form with pole of order at most m.

By parallel arguments as in [39, Proposition 20], the assertions in Remark 2.2 hold analogously for weakly holomorphic quasimodular forms. In particular, f_p is weakly holomorphic modular with pole of order at most m. The space of weakly holomorphic modular forms is generated by $\frac{1}{\Delta}$ over Mod, where

$$\Delta(q) = q \prod_{n \geq 1} (1 - q^n)^{24}$$

is the modular discriminant.[6] As a consequence,

$$f_p \in \frac{1}{\Delta^m} \mathsf{Mod}$$

and since f_p is of weight $k - 2p$ (and there are no non-zero modular forms of negative weight) we have $k \geq 2p - 12m$.

For quasimodular forms we include the following observation.

Lemma 2.4. The space of weakly holomorphic quasimodular forms with pole of order at most m is given by

$$\frac{1}{\Delta^m} \mathsf{QMod}\,.$$

Proof. Let f be a weakly holomorphic form with pole of order at most m and weight k and let

$$\widehat{f} = \sum_{r=0}^{p} f_r y^{-r}\,,$$

with $f = f_0$. Multiplying by Δ^m we have for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$

$$(\Delta^m \widehat{f})|_{k+12m}\gamma = (\Delta^m)|_{12m}\gamma \cdot (\widehat{f})|_k\gamma = \Delta^m \widehat{f}\,.$$

Since each $\Delta^m f_r$ is holomorphic at $i\infty$ this proves

$$f \in \frac{1}{\Delta^m} \mathsf{QMod}\,.$$

Analogous argument shows that the quotient of any quasimodular form by Δ^m defines a weakly holomorphic quasimodular form with pole of order at most m. □

[6]See [14] where the authors examine an explicit basis of the space of weakly holomorphic modular forms.

2.2 Hecke operators

Let $m \in \mathbb{N}$ and consider the set of integral matrices of determinant m

$$H_m = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}\,, ad - bc = m \right\} .$$

The modular group $\mathrm{SL}_2(\mathbb{Z})$ acts on H_m by left multiplication. The classical *Hecke operators* T_m acting on modular forms f of weight k are defined by [39, Section 4.1]

$$\mathsf{T}_m f = m^{k/2-1} \sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \backslash H_m} f|_k \gamma\,.$$

This definition is equivalent to [22, Ch. 3, Proposition 38]

$$\mathsf{T}_m = \sum_{ad=m} a^{k-1} \mathsf{B}_a \mathsf{U}_d\,. \tag{3.7}$$

The action of (3.7) naturally extends to the action of the q-expansion of weakly holomorphic quasimodular forms. We prove that the action again defines a weakly holomorphic quasimodular form. For simplicity (we will only use this case) we restrict to the case when f has a pole of order at most one.

Lemma 2.5. Let $f \in \frac{1}{\Delta}\mathsf{QMod}$ be of weight k. Then $\mathsf{T}_m f$ is a weakly holomorphic quasimodular form of weight k with pole of order at most m, i.e.

$$\mathsf{T}_m f \in \frac{1}{\Delta^m}\mathsf{QMod}\,.$$

Proof. In [31] it is shown that T_m defines a map $\mathsf{QMod} \to \mathsf{QMod}$ preserving the weight. We briefly recall the key arguments for $f \in \mathsf{QMod}$. The definition of quasimodular forms is equivalent to the condition[7]

$$(f|_k\gamma)(\tau) = \sum_{r=0}^{p} \left(\frac{c}{c\tau + d}\right)^r f_r(\tau) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})\,,$$

where f_r are as in Definition 2.1. Defining a modification of the slash operator for quasimodular forms[8]

$$(f||_k A)(\tau) = \sum_{r=0}^{p} (-c)^r (c\tau+d)^r (f_r|_k A)(\tau) \text{ for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})\,,$$

[7] This notion is called 'differential modular form' in [31]. As pointed out in [39, Section 5.3], this notion is equivalent to be a quasimodular form.

[8] This definition differs from [31, Equation 12] by a factor m^{-p}, where p is the depth of f. Our definition of the Hecke operator differs by the same factor.

then the quasimodularity is equivalent to

$$f||_k\gamma = f \text{ for all } \gamma \in \mathrm{SL}_2(\mathbb{Z})\,.$$

This leads to a parallel treatment of Hecke operators as in the classical context of modular forms. By [31, Proposition 2] we have

$$f||_k(\gamma A) = f||_k A\,, \text{ for all } \gamma \in \mathrm{SL}_2(\mathbb{Z})\,, A \in \mathrm{GL}_2^+(\mathbb{R})$$

and we define

$$\mathsf{T}_m f = m^{k/2-1} \sum_{A \in \mathrm{SL}_2(\mathbb{Z})\backslash H_m} f||_k A\,.$$

This definition is then independent of a choice of representatives of $\mathrm{SL}_2(\mathbb{Z})\backslash H_m$. To conclude that $\mathsf{T}_m f$ is a quasimodular form, we would like to argue that it is invariant under $(-)||_k\gamma$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. This statement, however, is not sensible at the moment[9] because the definition of $(-)||_k\gamma$ relies on the existence of associated functions f_r. This technicality is resolved in [31, Section 2.4, 2.5] by considering a certain period domain $\mathcal{P}$ and identifying quasimodular forms as holomorphic functions on $\mathcal{P}$, which are left $\mathrm{SL}_2(\mathbb{Z})$-invariant and satisfy a transformation property for a right action of the subgroup of upper triangular matrices. The domain $\mathcal{P}$ is contained in $\mathrm{GL}_2(\mathbb{C})$ and it contains the upper-half plane $\mathbb{H}$. The actions are given by left resp. right multiplication. The argument carries over to weakly holomorphic quasimodular forms without change.

A particular set of representatives for $\mathrm{SL}_2(\mathbb{Z}) \setminus H_m$ is given by

$$\left\{ \gamma_b = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{N}, ad = m, 0 \le b < d \right\}\,.$$

Note that $(-)||_k\gamma_b = (-)|_k\gamma_b$ because the terms for $r > 0$ vanish. Since

$$\mathsf{U}_d f(\tau) = \frac{1}{d} \sum_{0 \le b < d} f\left(\frac{\tau + b}{d}\right)\,,$$

we thus recover equation (3.7):

$$\begin{aligned} \mathsf{T}_m f(\tau) &= m^{k/2-1} \sum_{\substack{ad=m \\ 0 \le b < d}} d^{-k} m^{k/2} f\left(\frac{a\tau + b}{d}\right) \\ &= \sum_{ad=m} a^{k-1} \mathsf{B}_a \mathsf{U}_d f(\tau)\,. \end{aligned}$$

[9]We are grateful to the referee for pointing out this subtle detail.

For weakly holomorphic quasimodular forms $f \in \frac{1}{\Delta}\mathsf{QMod}$ we follow the same proof. The difference here is that the functions f_r are allowed to have simple poles at $i\infty$. The slash operator $(-)||_k$ however may turn a simple pole into a pole of higher order. For $(-)||_k\gamma_b$ this order is bounded by m. As a consequence, $\mathsf{T}_m f$ is weakly holomorphic quasimodular with pole of order at most m. □

For our study of the multiple cover formula in Section 3 we will require a more flexible notion, where the exponent is not necessarily related to the weight. The action of this operator will preserve the weight of weakly holomorphic quasimodular forms, it will, however, introduce poles and level structure.

Definition 2.6. For $\ell \in \mathbb{Z}$, we define

$$\mathsf{T}_{m,\ell} = \sum_{ad=m} a^{\ell-1} \mathsf{B}_a \mathsf{U}_d \,.$$

The operator $\mathsf{T}_{m,\ell}$ is simply the m-th Hecke operator of weight ℓ, which we let act on functions of weight k. By Möbius inversion we may rewrite each of them in terms of the other (see [1, Section 2.7]). For this, let μ be the Möbius function.

Lemma 2.7. The action of $\mathsf{T}_{m,\ell}$ on weakly holomorphic quasimodular forms of weight k is given by

$$\mathsf{T}_{m,\ell} = \sum_{ad=m} c_{k,\ell}(a) \mathsf{B}_a \mathsf{T}_d \,,$$

where

$$c_{k,\ell}(a) = \sum_{r|a} r^{\ell-1} \mu\left(\frac{a}{r}\right) \left(\frac{a}{r}\right)^{k-1} .$$

Proof. The formula for $c_{k,\ell}$ above can be rewritten as

$$c_{k,\ell} = \mathrm{Id}_{\ell-1} \star (\mu \cdot \mathrm{Id}_{k-1}) \,,$$

where $\mathrm{Id}_{\ell-1}(n) = n^{\ell-1}$ is the $(\ell-1)$-th power function and $\star$ denotes Dirichlet convolution, i.e. for functions g, h we have

$$(g \star h)(m) = \sum_{ad=m} g(a)h(d) \,.$$

Note also that B is multiplicative with respect to composition, i.e. for $e \mid a$ we have $\mathsf{B}_a = \mathsf{B}_e \mathsf{B}_{\frac{a}{e}}$ and therefore

$$\begin{aligned}
\mathsf{T}_{m,\ell} &= \sum_{ad=m} a^{\ell-1} \mathsf{B}_a \mathsf{U}_d \\
&= \sum_{ad=m} \big(\mathrm{Id}_{\ell-1} \star (\mu \cdot \mathrm{Id}_{k-1}) \star \mathrm{Id}_{k-1}\big)(a) \mathsf{B}_a \mathsf{U}_d \\
&= \sum_{ad=m} \left(\sum_{e|a} c_{k,\ell}(e) \left(\frac{a}{e}\right)^{k-1} \right) \mathsf{B}_a \mathsf{U}_d \\
&= \sum_{uw=m} c_{k,\ell}(u) \mathsf{B}_u \left(\sum_{v|w} v^{k-1} \mathsf{B}_v \mathsf{U}_{\frac{w}{v}} \right) \\
&= \sum_{uw=m} c_{k,\ell}(u) \mathsf{B}_u \mathsf{T}_w \,.
\end{aligned}$$

□

As a consequence we obtain the following result. Here, we let $\mathsf{Mod}(m)$ and $\mathsf{QMod}(m)$ be the space of modular resp. quasimodular forms for the congruence subgroup $\Gamma_0(m) \subset \mathrm{SL}_2(\mathbb{Z})$, see the introduction.

Proposition 2.8. Let $f \in \frac{1}{\Delta}\mathsf{QMod}$ be of weight k, then $\mathsf{T}_{m,\ell} f$ is a weakly holomorphic quasimodular of weight k with pole of order at most m for the congruence subgroup $\Gamma_0(m) \subset \mathrm{SL}_2(\mathbb{Z})$

$$\mathsf{T}_{m,\ell} f \in \frac{1}{\Delta^m} \mathsf{QMod}(m) \,.$$

Proof. We use the formula in Lemma 2.7 and treat each summand separately. By Lemma 2.4 each $\mathsf{T}_d f$ satisfies

$$\mathsf{T}_d f \in \frac{1}{\Delta^d} \mathsf{QMod} \,.$$

The action of B_a raises $q \mapsto q^a$, or equivalently $\tau \mapsto a\tau$, so it maps QMod to $\mathsf{QMod}(a)$, see [22, Ch. 3, Proposition 17]. Therefore

$$\mathsf{B}_a \mathsf{T}_d f \in \frac{1}{\Delta(q^a)^d} \mathsf{QMod}(a) \,.$$

Finally, the weakly holomorphic modular form for $\Gamma_0(a)$ defined by

$$\frac{\Delta(q)^a}{\Delta(q^a)}$$

is in fact holomorphic at $i\infty$, i.e. contained in $\mathsf{Mod}(a)$. Hence the same is true for its d-th power and we find

$$\mathsf{B}_a \mathsf{T}_d f \in \frac{1}{\Delta^m} \mathsf{QMod}(a)\,.$$

which concludes the proof since $\mathsf{QMod}(a) \subset \mathsf{QMod}(m)$. ☐

For later reference, we list the following basic commutator relations between the above operators acting on weakly holomorphic quasimodular forms f of weight k. Recall, that the algebra $\mathsf{QMod}(m)$ is freely generated by the Eisenstein series C_2 over the algebra $\mathsf{Mod}(m)$ of modular forms. Formal differentiation with respect to C_2 is therefore well-defined.

Lemma 2.9. Let $d,\, e \in \mathbb{N}$ and $\ell \in \mathbb{Z}$, then

(i) $\mathsf{B}_d \mathsf{B}_e = \mathsf{B}_{de} = \mathsf{B}_e \mathsf{B}_d\,,$

(ii) $\mathsf{U}_d \mathsf{U}_e = \mathsf{U}_{de} = \mathsf{U}_e \mathsf{U}_d\,,$

(iii) $\mathsf{D}_q \mathsf{B}_d = d\, \mathsf{B}_d \mathsf{D}_q\,, \quad \mathsf{U}_d \mathsf{D}_q = d\, \mathsf{D}_q \mathsf{U}_d\,,$

(iv) $\mathsf{T}_{m,\ell+2} \mathsf{D}_q = m\, \mathsf{D}_q \mathsf{T}_{m,\ell}\,,$

(v) $\frac{d}{dC_2} \mathsf{T}_{m,\ell+2} = m\, \mathsf{T}_{m,\ell} \frac{d}{dC_2}\,,$

(vi) $[\frac{d}{dC_2}, \mathsf{D}_q] = -2k.$

Proof. The proof for (i)-(iv) follows directly from the definition. For (v) one may use that under the isomorphism $\widehat{f} \mapsto f$ the differentiation $\frac{d}{dC_2}$ corresponds to differentiation with respect to $\frac{1}{8\pi y}$, see Remark 2.2. The statement (v) is then checked as an identity of Laurent series in q with polynomial coefficients in y^{-1}. The commutator relation (vi) is well-known, see e.g. [39, Section 5.3].

☐

3 Multiple cover formula

This section contains a discussion of the multiple cover formula. We start by recalling the conjecture formulated in [32]. Then, we study the conjecture for the descendent potentials associated to elliptic K3 surfaces. The result is expressed in terms of Hecke operators. The discussion naturally leads to a candidate for the holomorphic anomaly equation in higher divisibility. We conclude with a proof of the multiple cover formula in fiber direction.

3.1 Multiple cover formula

Let S be a nonsingular projective K3 surface, $\beta \in H_2(S,\mathbb{Z})$ be a *primitive* effective curve class, $m \in \mathbb{N}$ and $d \mid m$ be a divisor of m. The proposed formula by Oberdieck and Pandharipande involves a choice of a real isometry

$$\varphi_{d,m} \colon \Big(H^2(S,\mathbb{R})\,,\langle\;,\;\rangle\Big) \to \Big(H^2(S_d,\mathbb{R})\,,\langle\;,\;\rangle\Big)$$

between two K3 surfaces such that

$$\varphi_{d,m}\left(\frac{m}{d}\beta\right) \in H_2(S_d,\mathbb{Z})$$

is a primitive effective curve class[10]. In [9] the second author proved that such an isometry can always be found and Gromov–Witten invariants are in fact independent of the choice of isometry.

Consider integers $a_i \in \mathbb{N}$, cohomology classes $\gamma_i \in H^*(S,\mathbb{Q})$ and let $\deg = \sum \deg(\gamma_i)$. Then, the conjectured multiple cover formula [32, Conjecture C2], identical to Conjecture 1.4 in Section 1, is

$$\begin{aligned}&\big\langle \tau_{a_1}(\gamma_1)\dots\tau_{a_n}(\gamma_n)\big\rangle_{g,m\beta}\\ &= \sum_{d\mid m} d^{2g-3+\deg}\big\langle \tau_{a_1}(\varphi_{d,m}(\gamma_1))\dots\tau_{a_n}(\varphi_{d,m}(\gamma_n))\big\rangle_{g,\varphi_{d,m}\left(\frac{m}{d}\beta\right)}\,.\end{aligned}$$

Let S be an elliptic K3 surface with a section[11]. The full (reduced) Gromov–Witten theory of K3 surfaces is captured by S with curve class $mB+hF$ via standard deformation arguments using the Torelli theorem. In fact, the multiple cover conjecture can be captured entirely via S as well: we may choose *the same* $S_d = S$ for any d dividing m and h. For $l \in \mathbb{Q}^*$ we define

$$\phi_l \colon H^*(S,\mathbb{Q}) \to H^*(S,\mathbb{Q})$$

acting on $U = \mathbb{Q}\langle F, W\rangle$ as

$$\phi_l(F) = \frac{1}{l}F\,, \qquad \phi_l(W) = lW\,,$$

and trivially on the orthogonal complement $U^\perp$. For $d \mid m$ and $d \mid h$ we may choose $\varphi_{d,m}$ as $\phi_{\frac{d}{m}}$:

$$\phi_{\frac{d}{m}}\left(\frac{m}{d}B + \frac{h}{d}F\right) = B + \left(\frac{m(h-m)}{d^2}+1\right)F \;\text{ in } H_2(S,\mathbb{Z})$$

[10] We view curve classes also as cohomology classes under the natural isomorphism $H_2(S,\mathbb{Z}) \cong H^2(S,\mathbb{Z})$.

[11] Notations here are as in Section 1. In particular, we use the modified degree function $\underline{\deg}$.

which is a primitive curve class.

Altering the curve class via the isometry ϕ therefore results in additional factors of $\frac{d}{m}$ or $\frac{m}{d}$ while keeping the descendent insertions unchanged. This explains the change in exponents

$$2g-3+\deg \longleftrightarrow 2g-3+\underline{\deg}$$

and the factor $m^{\deg-\underline{\deg}}$ in the multiple cover formula below for the descendent potential. We use the operator $\mathsf{T}_{m,\ell}$ introduced in Definition 2.6. As pointed out in Section 1.4, this is the m-th Hecke operator for functions of weight ℓ, which we let act on $\mathsf{F}_{g,1}$ (which has weight $2g-12+\underline{\deg}$). Before stating the conjecture, we want to discuss the role of tautological classes and compatibility with respect to restriction to boundary strata.

3.2 Compatibility I

We will find it convenient to use pullbacks of tautological classes from $\overline{M}_{g,n}$ instead of ψ-classes on $\overline{M}_{g,n}(S,\beta)$. For $2g-2+n>0$, let

$$R^*(\overline{M}_{g,n}) \subseteq A^*(\overline{M}_{g,n})$$

be the tautological ring of $\overline{M}_{g,n}$. For a tautological class $\alpha \in R^*(\overline{M}_{g,n})$, we consider the invariants

$$\big\langle \alpha; \gamma_1, \ldots, \gamma_n \big\rangle = \int_{[\overline{M}_{g,n}(S,\beta)]^{red}} \pi^*\alpha \cup \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i)\,,$$

where $\pi\colon \overline{M}_{g,n}(S,\beta) \to \overline{M}_{g,n}$ is the stabilization morphism. We write

$$\mathsf{F}_{g,m}\big(\alpha; \gamma_1, \ldots, \gamma_n\big) = \sum_{h \geq 0} \big\langle \alpha; \gamma_1, \ldots, \gamma_n \big\rangle_{g,mB+hF}\, q^{h-m}$$

for the generating series in divisibility m. By the usual trading of cotangent line classes, these generating series are related to the ones defined via cotangent classes on $\overline{M}_{g,n}(S,\beta)$. Any monomial in ψ- and κ-classes can be written, after adding markings, as a product of ψ-classes. This procedure leaves deg and $\underline{\deg}$ unchanged. Before stating the multiple cover formula below, we explain the compatibility with respect to restriction to boundary strata in $\overline{M}_{g,n}(S,\beta)$.

A crucial point for this compatibility is the splitting behavior of the reduced class. Consider the pullback of the boundary divisor

$$\overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$$

under the stabilization morphism π. Let α be the pushforward of a tautological class (we will omit pushforwards in the notation below). By the restriction property of the reduced class, we obtain

$$\mathsf{F}_{g,m}\big(\alpha;\gamma\big) = \mathsf{F}_{g-1,m}\big(\alpha;\gamma\Delta_S\big)\,.$$

Then, the compatibility follows from two facts. Firstly, for the diagonal class Δ_S we have

$$\big(\deg - \underline{\deg}\big)(\Delta_S) = 0\,,$$

thus the factor $m^{\deg - \underline{\deg}}$ in Conjecture 3.1 below remains unchanged. Secondly, we have $\underline{\deg}(\Delta_S) = 2$ which precisely offsets the genus reduction from g to $g-1$ in the formula

$$\ell = 2g - 2 + \underline{\deg}\,.$$

Next, consider the pullback of the boundary divisor

$$\overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g,n}$$

under the stabilization morphism π. Let

$$\alpha = \alpha_1 \boxtimes \alpha_2\,, \quad \{1,\ldots,n\} = I_1 \cup I_2\,, \quad \gamma = \gamma_1 \boxtimes \gamma_2$$

be the pushforward of the product of tautological classes, the splitting of markings, and the splitting of the insertions respectively. The Künneth decomposition of the class of the diagonal is denoted by

$$[\Delta_S] = \sum_j \Delta_j \boxtimes \Delta^j\,.$$

The splitting property implies that

$$\mathsf{F}_{g,m}\big(\alpha;\gamma\big) = \sum_{m_1+m_2=m} \sum_j \Bigg(\mathsf{F}_{g_1,m_1}\big(\alpha_1;\gamma_{I_1}\Delta_j\big) \cdot \mathsf{F}^{vir}_{g_2,m_2}\big(\alpha_1;\gamma_{I_1}\Delta^j\big) + \mathsf{F}^{vir}_{g_1,m_1}\big(\alpha_1;\gamma_{I_1}\Delta_j\big) \cdot \mathsf{F}_{g_2,m_2}\big(\alpha_1;\gamma_{I_1}\Delta^j\big) \Bigg)\,.$$

The virtual class for non-zero curve classes vanishes, thus the contribution F^{vir} is a number. As a consequence, no non-trivial products of generating series appear when we use boundary expressions. By similar consideration as above, using the deg and $\underline{\deg}$ for the diagonal class, we find that the multiple cover formula is compatible with respect to this boundary divisor as well. We can now state the multiple cover formula for the generating series with tautological classes:

Conjecture 3.1. For $\underline{\deg}$-homogeneous $\gamma_i \in H^*(S,\mathbb{Q})$,

$$\mathsf{F}_{g,m}\big(\alpha;\gamma_1,\dots,\gamma_n\big) = m^{\deg - \underline{\deg}}\,\mathsf{T}_{m,\ell}\left(\mathsf{F}_{g,1}\big(\alpha;\gamma_1,\dots,\gamma_n\big)\right),$$

where $\deg = \sum \deg(\gamma_i)$, $\underline{\deg} = \sum \underline{\deg}(\gamma_i)$ and $\ell = 2g-2+\underline{\deg}$.

Based on the discussion above, the same formula is conjectured for the potential

$$\mathsf{F}_{g,m}\big(\tau_{a_1}(\gamma_1)\dots\tau_{a_n}(\gamma_n)\big)\,.$$

We now show that our presentation of the multiple cover formula is equivalent to the original formula.

Lemma 3.2. Conjecture 1.4 for all $d \mid m$ is equivalent to Conjecture 3.1 for m.

Proof. By the deformation invariance of the reduced class, the Gromov–Witten invariants for arbitrary curve classes are fully captured by an elliptic K3 surface with a section. The primitive curve classes are $B + hF \in H_2(S,\mathbb{Z})$. Taking the coefficient of q^{mh-m} in Conjecture 3.1 gives a multiple cover formula for the curve class $mB + mhF$ which matches the formula in Conjecture 1.4. It is the other implication which we have to justify.

The generating series $\mathsf{F}_{g,m}$ involves curve classes $mB + hF$ of different divisibilities bounded by m. We apply Conjecture 1.4 to each invariant and use the isometries ϕ. Note that each appearance of $\gamma_i = F$ introduces a factor of $\frac{m}{d}$, while each appearance of $\gamma_i = W$ gives $\frac{d}{m}$. Moreover,

$$|\{i \mid \gamma_i = F\}| - |\{i \mid \gamma_i = W\}| = \deg - \underline{\deg}\,,$$

and therefore

$$\begin{aligned}
\mathsf{F}_{g,m}\big(\alpha;\gamma_1,\dots,\gamma_n\big) &= \sum_{h\geq 0} \big\langle \alpha;\gamma_1,\dots,\gamma_n\big\rangle_{g,mB+hF}\, q^{h-m} \\
&= \sum_{h\geq 0}\sum_{\substack{d\mid m\\ d\mid h}} d^{2g-3+\deg}\left(\frac{m}{d}\right)^{\deg-\underline{\deg}} \big\langle \alpha;\gamma_1,\dots,\gamma_n\big\rangle_{g,B+\left(\frac{m(h-m)}{d^2}+1\right)F}\, q^{h-m} \\
&= m^{\deg-\underline{\deg}} \sum_{d\mid m} d^{2g-3+\underline{\deg}} \left(\sum_{h\geq 0} \big\langle \alpha;\gamma_1,\dots,\gamma_n\big\rangle_{g,B+\left(\frac{m}{d}(h-\frac{m}{d})+1\right)F}\, \left(q^d\right)^{h-\frac{m}{d}} \right) \\
&= m^{\deg-\underline{\deg}} \sum_{d\mid m} d^{2g-3+\underline{\deg}} \left(\mathsf{B}_d \mathsf{U}_{\frac{m}{d}} \sum_{h\geq 0} \big\langle \alpha;\gamma_1,\dots,\gamma_n\big\rangle_{g,B+hF}\, q^{h-1} \right) \\
&= m^{\deg-\underline{\deg}} \sum_{d\mid m} d^{2g-3+\underline{\deg}}\, \mathsf{B}_d \mathsf{U}_{\frac{m}{d}} \mathsf{F}_{g,1}\big(\alpha;\gamma_1,\dots,\gamma_n\big) \\
&= m^{\deg-\underline{\deg}}\, \mathsf{T}_{m,\ell}\left(\mathsf{F}_{g,1}\big(\alpha;\gamma_1,\dots,\gamma_n\big)\right). \qquad \square
\end{aligned}$$

As a direct consequence, the multiple cover formula implies level m quasimodularity.

Proposition 3.3. If the generating series $\mathsf{F}_{g,m}$ satisfies the multiple cover formula, it satisfies the quasimodularity conjecture. More precisely,

$$\mathsf{F}_{g,m} \in \frac{1}{\Delta(q)^m}\mathsf{QMod}(m)\,.$$

Proof. The descendent potentials for primitive curve classes are weakly holomorphic quasimodular with pole of order at most 1 and weight $2g-12+\underline{\deg}$, see [29, Theorem 4] and [8, Theorem 9]. The claim thus follows from Proposition 2.8. □

3.3 Multiple cover formula in fiber direction

When the curve class is a multiple of the fiber class F, the multiple cover formula reduces to a property of the Gromov–Witten invariant of elliptic curves. Relevant properties are conjectured in [38].

Let $S \to \mathbb{P}^1$ be an elliptic K3 surface with section and let $\beta = mF$. By Section 7, Case 1, we may assume at least one of the insertions is the point class $\gamma_1 = \mathsf{p}$ and $g \geq 1$. Let

$$\iota\colon E \hookrightarrow S$$

be the inclusion of a fiber, representing the class F. Since the point class is represented by a transverse intersection of E and the section B, the Gromov–Witten theory of S localizes to the Gromov–Witten theory of E with the curve class mE. Computation of the obstruction bundle shows that the invariant is of the form

$$\big\langle \tau_{a_1}(\mathsf{p})\tau_{a_2}(\gamma_2)\ldots\tau_{a_n}(\gamma_n)\big\rangle^S_{g,mF} = \big\langle \lambda_{g-1};\tau_{a_1}(\omega)\tau_{a_2}(\iota^*\gamma_2)\ldots\tau_{a_n}(\iota^*\gamma_n)\big\rangle^E_{g,mE}$$

where $\lambda_{g-1} = c_{g-1}(\mathbb{E}_g)$. In particular, if $\gamma_i \in \mathbb{Q}\langle F\rangle \oplus U^\perp \oplus \mathbb{Q}\langle \mathsf{p}\rangle$, the invariant vanishes. Consider the following generating series

$$\mathsf{F}^E_g\big(\tau_{a_1}(\gamma_1)\ldots\tau_{a_n}(\gamma_n)\big) = \sum_{m\geq 0}\big\langle\lambda_{g-1};\tau_{a_1}(\gamma_1)\ldots\tau_{a_n}(\gamma_n)\big\rangle^E_{g,mE}\,q^m$$

where $\gamma_i = 1$ or ω and $\sum a_i + \sum \deg(\gamma_i) = g-1+n$.

The generating series F^E_g has a simple description in terms of Eisenstein series. The following formula is conjectured in [38].

Lemma 3.4. For $g \geq 1$,

$$\mathsf{F}^E_g\big(\tau_{g-1}(\omega)\big) = \frac{g!}{2^{g-1}}C_{2g}\,.$$

Proof. In [38, Proposition 4.4.7] this formula is given under assuming the Virasoro constraint for $\mathbb{P}^1 \times E$. The Virasoro constraint for any toric bundle over a nonsingular variety which satisfies the Virasoro constraint is proven in [13]. Combining this result with the Virasoro constraint for elliptic curves [35], the result follows. □

When $\beta = mF$, Conjecture 1.4 is equivalent to the following proposition.

Proposition 3.5. There exists $c \in \mathbb{Q}$ such that

$$\mathsf{F}_g^E\big(\tau_{a_1}(\omega)\dots\tau_{a_r}(\omega)\tau_{a_{r+1}}(1)\dots\tau_{a_{r'}}(1)\big) = c\,\mathsf{D}_q^{r-1}\mathsf{F}_g^E\big(\tau_{g-1}(\omega)\big)\,.$$

Proof. Boundary strata with a vertex of genus less than g do not contribute because the invariants involve λ_h vanishes on $\overline{M}_{g,n}(E,m)$ when $h \geq g$. If $r' > r$, then $\sum a_i \geq g$ and we can reduce to the case when $r' = r$ by the topological recursion on the ψ-monomial in $R^{\geq g}(\overline{M}_{g,n})$ [23]. If $r' = r$, then $\sum a_i = g-1$ and similar argument as in Section 7, Case 3 can be applied. Therefore F_g^E is proportional to

$$\mathsf{F}_g^E\big(\tau_{g-1}(\omega)\tau_0(\omega)^{r-1}\big) = \mathsf{D}_q^{r-1}\mathsf{F}_g^E\big(\tau_{g-1}(\omega)\big)$$

where the equality comes from the divisor equation. □

Remark 3.6. One can find a closed formula for the constant $c \in \mathbb{Q}$ by integrating tautological classes on $\overline{M}_{g,n}$.

4 Holomorphic anomaly equation

This section contains a proof of Proposition 1.5. We derive the holomorphic anomaly equation for $m \geq 1$ from the conjectural multiple cover formula, such that both are compatible[12]. It turns out that the equation is almost identical to the one in the primitive case. Additional factors appear only in the last two terms, which are specific to K3 surfaces. We refer to [34, Section 7.3] for explanations on the appearance of these terms.

Proof of Proposition 1.5. Let $\gamma_1, \dots, \gamma_n \in H^*(S)$ with

$$\deg = \sum_i \deg(\gamma_i)\,, \quad \underline{\deg} = \sum_i \underline{\deg}(\gamma_i)\,.$$

We will simply write γ to denote $\gamma_1, \dots, \gamma_n$. Assume that the multiple cover formula (3.4) holds for all divisors $d \mid m$ and all descendent insertions. Using

[12] We should point out that this derivation should be lifted to the cycle-valued holomorphic anomaly equation. Tautological classes play no role here.

Lemma 3.2, also Conjecture 3.1 holds. By Proposition 3.3, the descendent potentials are quasimodular forms of level m and we can consider the $\frac{d}{dC_2}$-derivative. We apply the $\frac{d}{dC_2}$-derivative to Conjecture 3.1 and use the commutator relations Lemma 2.9 to obtain:

$$\begin{aligned}\frac{d}{dC_2}\mathsf{F}_{g,m}(\alpha;\gamma) &= \frac{d}{dC_2}\Big(m^{\deg-\underline{\deg}}\mathsf{T}_{m,2g-2+\underline{\deg}}\mathsf{F}_{g,1}(\alpha;\gamma)\Big)\\ &= m^{\deg-\underline{\deg}+1}\mathsf{T}_{m,2g-4+\underline{\deg}}\frac{d}{dC_2}\mathsf{F}_{g,1}(\alpha;\gamma)\,.\end{aligned}$$

We want to explain that the last row precisely recovers the definition of $\mathsf{H}_{g,m}$ in (3.1), after applying the holomorphic anomaly equation for the primitive series [33, Theorem 4]:

$$\frac{d}{dC_2}\mathsf{F}_{g,1}(\alpha;\gamma) = \mathsf{H}_{g,1}(\alpha;\gamma)\,.$$

We do so by explaining how each term of $\mathsf{H}_{g,1}(\alpha;\gamma)$ is affected:

(i) The degree deg of $\mathsf{F}_{g-1,1}(\alpha;\gamma\,\Delta_{\mathbb{P}^1})$ has increased by one. The genus, however, dropped by 1. Thus, the first term precisely matches the multiple cover formula, i.e.

$$\mathsf{F}_{g-1,m}(\alpha;\gamma\Delta_{\mathbb{P}^1}) = m^{\deg-\underline{\deg}+1}\mathsf{T}_{m,2g-4+\underline{\deg}}\Big(\mathsf{F}_{g-1,1}(\alpha;\gamma\,\Delta_{\mathbb{P}^1})\Big)\,.$$

(ii) The virtual class is non-zero only for curve class $\beta = 0$ and genus 0, 1, see Section 1. In these cases, the potential $\mathsf{F}^{vir}_{g_2}$ is simply a number and the operator $\mathsf{T}_{m,\ell}$ acts non-trivially only on $\mathsf{F}_{g_1,m}$. We distinguish the two cases:

$g_2 = 0$. The virtual class is given by the fundamental class and the integral is given by intersection pairing on S. Non-trivial terms are obtained from $\delta_i^\vee = 1$ or F. If $\delta_i^\vee = 1$ then

$$\deg(\gamma_{I_2}) = \underline{\deg}(\gamma_{I_2}) = 2\,.$$

The modified degree $\underline{\deg}$ of $\mathsf{F}_{g_1,1}(\alpha_{I_1};\gamma_{I_1}\delta_i)$ has decreased by 2, whereas deg decreased by 1 (the insertion $\delta_i = F$ contributes deg = 1). The term thus matches the multiple cover formula:

$$\begin{aligned}&\mathsf{F}_{g_1,m}(\alpha_{I_1};\gamma_{I_1}\delta_i)\\ &= m^{\deg-\underline{\deg}+1}\mathsf{T}_{m,2g-4+\underline{\deg}}\Big(\mathsf{F}_{g_1,1}(\alpha_{I_1};\gamma_{I_1}\delta_i)\Big)\,.\end{aligned}$$

If $\delta_i^\vee = F$ then

$$\deg(\gamma_{I_2}) = 1\,,\quad \underline{\deg}(\gamma_{I_2}) = 2\,.$$

The modified degree $\underline{\deg}$ of $\mathsf{F}_{g_1,1}\big(\alpha_{I_1};\gamma_{I_1}\delta_i\big)$ has decreased by 2, whereas deg decreased by 1. The term matches the multiple cover formula.

$g_2 = 1$. The virtual class is given by $c_2(S)$ and the integral is given by intersection pairing on S. Non-trivial terms are obtained only from $\delta_i^\vee = 1$ and

$$\deg(\gamma_{I_2}) = \underline{\deg}(\gamma_{I_2}) = 0\,.$$

Analogously to case (i), the degree deg of $\mathsf{F}_{g_1,1}\big(\alpha_{I_1};\gamma_{I_1}\delta_i\big)$ has increased by 1, $\underline{\deg}$ remained unchanged, and the genus dropped by 1. The term matches the multiple cover formula.

(iii) The modified degree $\underline{\deg}$ of $\mathsf{F}_{g,1}\big(\alpha\psi_i;\gamma_1,\ldots,\pi^*\pi_*\gamma_i,\ldots,\gamma_n\big)$ has decreased by 2, whereas deg decreased by 1. Again we find that the term matches the multiple cover formula

$$\begin{aligned}&\mathsf{F}_{g,m}\big(\alpha\psi_i;\gamma_1,\ldots,\pi^*\pi_*\gamma_i,\ldots,\gamma_n\big)\\ &= m^{\deg-\underline{\deg}+1}\mathsf{T}_{m,2g-4+\underline{\deg}}\Big(\mathsf{F}_{g,1}\big(\alpha\psi_i;\gamma_1,\ldots,\pi^*\pi_*\gamma_i,\ldots,\gamma_n\big)\Big)\,.\end{aligned}$$

(iv) The degree of $\langle\gamma_i,F\rangle\mathsf{F}_{g,1}\big(\alpha;\gamma_1,\ldots,F,\ldots,\gamma_n\big)$ remains unchanged, whereas $\underline{\deg}$ decreased by 2. An additional factor of $\frac{1}{m}$ therefore appears:

$$\begin{aligned}&\frac{1}{m}\langle\gamma_i,F\rangle\mathsf{F}_{g,m}\big(\alpha;\gamma_1,\ldots,F,\ldots,\gamma_n\big)\\ &= m^{\deg-\underline{\deg}+1}\mathsf{T}_{m,2g-4+\underline{\deg}}\Big(\langle\gamma_i,F\rangle\mathsf{F}_{g,1}\big(\alpha;\gamma_1,\ldots,F,\ldots,\gamma_n\big)\Big)\,.\end{aligned}$$

(v) The term $\mathsf{F}_{g,1}\big(\ldots,\sigma_1(\gamma_i,\gamma_j),\ldots,\sigma_2(\gamma_i,\gamma_j),\ldots\big)$ is similar to the previous case: deg remains unchanged, whereas $\underline{\deg}$ decreases by 2, giving rise to an additional factor of $\frac{1}{m}$:

$$\begin{aligned}&\frac{1}{m}\mathsf{F}_{g,m}\big(\gamma_1,\ldots,\sigma_1(\gamma_i,\gamma_j),\ldots,\sigma_2(\gamma_i,\gamma_j),\ldots,\gamma_n\big)\\ &= m^{\deg-\underline{\deg}+1}\mathsf{T}_{m,2g-4+\underline{\deg}}\Big(\mathsf{F}_{g,1}\big(\gamma_1,\ldots,\sigma_1(\gamma_i,\gamma_j),\ldots,\sigma_2(\gamma_i,\gamma_j),\ldots,\gamma_n\big)\Big)\end{aligned}$$

We arrive at the level m holomorphic anomaly equation (3.1) which appeared in Section 1. $\square$

4.1 Divisor equation

For primitive curve classes, it was pointed out in [33, Section 3.6, Case (i)] that the holomorphic anomaly equation in genus 0 is compatible with the divisor equation. For divisibility m, let

$$\frac{d}{d\gamma} = \langle \gamma, F\rangle \mathsf{D}_q + m\langle \gamma, W\rangle\,, \quad \gamma \in H^2(S)\,.$$

The divisor equation implies that

$$\begin{aligned}&\mathsf{F}_{g,m}\big(\tau_{a_1}(\gamma_1)\dots\tau_{a_{n-1}}(\gamma_{n-1})\tau_0(\gamma_n)\big)\\ &= \frac{d}{d\gamma_n}\mathsf{F}_{g,m}\big(\tau_{a_1}(\gamma_1)\dots\tau_{a_{n-1}}(\gamma_{n-1})\big)\\ &\quad + \sum_{i=1}^{n-1}\mathsf{F}_{g,m}\big(\tau_{a_1}(\gamma_1)\dots\tau_{a_i-1}(\gamma_i\cup\gamma_n)\dots\tau_{a_{n-1}}(\gamma_{n-1})\big)\,.\end{aligned}$$

The compatibility with the divisor equation corresponds to

$$\begin{aligned}&\mathsf{H}_{g,m}\big(\tau_{a_1}(\gamma_1)\dots\tau_{a_{n-1}}(\gamma_{n-1})\tau_0(\gamma_n)\big)\\ &= \frac{d}{d\gamma_n}\mathsf{H}_{g,m}\big(\tau_{a_1}(\gamma_1)\dots\tau_{a_{n-1}}(\gamma_{n-1})\big)\\ &\quad - 2k\mathsf{F}_{g,m}\big(\tau_{a_1}(\gamma_1)\dots\tau_{a_{n-1}}(\gamma_{n-1})\big)\\ &\quad + \sum_{i=1}^{n-1}\mathsf{H}_{g,m}\big(\tau_{a_1}(\gamma_1)\dots\tau_{a_i-1}(\gamma_i\cup\gamma_n)\dots\tau_{a_{n-1}}(\gamma_{n-1})\big)\,,\end{aligned} \tag{3.8}$$

where k is the weight of $\mathsf{F}_{g,m}\big(\tau_{a_1}(\gamma_1)\dots\tau_{a_{n-1}}(\gamma_{n-1})\big)$ and we have used the commutator relation

$$\left[\frac{d}{dC_2}, \mathsf{D}_q\right] = -2k\,.$$

The same check as in the primitive case works for arbitrary divisibility. This relies on the fact that the divisor equation for W is the same as applying the differential operator

$$\mathsf{D}_q = q\frac{d}{dq}$$

to the generating series. Indeed, for the curve class $\beta = mB + hF$,

$$\langle \beta, W\rangle = -2m + h + m = h - m\,,$$

which matches the exponent of q^{h-m} in the generating series $\mathsf{F}_{g,m}$. The divisor equation for F acts as multiplication by m on the generating series.

In Section 7, the refined induction reduces any generating series ultimately to genus 0 and 1. We thus have to justify compatibility of the holomorphic anomaly equation for generating series of the form

$$\mathsf{F}_{1,m}\big(\tau_0(\mathsf{p})\tau_0(\gamma_1)\ldots\tau_0(\gamma_n)\big)\,, \quad \gamma_i \in H^2(S)\,.$$

This compatibility however is true. By Proposition 6.2, the multiple cover formula, which is compatible with the divisor equation, holds in genus ≤ 1. Thus, we also find compatibility for the holomorphic anomaly equation.

Example 4.1. We consider $\mathsf{F}_{0,m}\big(\tau_0(W)^2\big)$ to illustrate the above compatibility. To compute $\mathsf{H}_{0,m}$, we use that $\sigma(W \boxtimes W) = U^\perp$, where the endomorphism σ is as defined in Section 1. Since the curve classes are contained in U, application of the divisor equation to a basis of $U^\perp$ implies

$$\mathsf{F}_{0,m}\big(\tau_0(U^\perp)\big) = 0\,.$$

We find that

$$\mathsf{H}_{0,m}\big(\tau_0(W)^2\big) = -4\mathsf{F}_{0,m}\big(\tau_1(1)\tau_0(W)\big) + \frac{40}{m}\mathsf{F}_{0,m}\big(\tau_0(F)\tau_0(W)\big)\,.$$

In the above notation, $\gamma_n = W$ is the second W and $k = -10$ is the weight of $\mathsf{F}_{0,m}\big(\tau_0(W)\big)$. We have to check that

$$\mathsf{H}_{0,m}\big(\tau_0(W)^2\big) = \mathsf{D}_q\mathsf{H}_{0,m}\big(\tau_0(W)\big) + 20\mathsf{F}_{0,m}\big(\tau_0(W)\big)\,.$$

By the dilaton equation, we can verify

$$\begin{aligned}
&\mathsf{H}_{0,m}\big(\tau_0(W)^2\big) - \mathsf{D}_q\mathsf{H}_{0,m}\big(\tau_0(W)\big)\\
&= -2\mathsf{D}_q\mathsf{F}_{0,m}\big(\tau_1(1)\big) - 4\mathsf{F}_{0,m}\big(\tau_0(W)\big) + \frac{20}{m}\mathsf{F}_{0,m}\big(\tau_0(F)\tau_0(W)\big)\\
&= 4\mathsf{D}_q\mathsf{F}_{0,m}\big(\emptyset\big) - 4\mathsf{D}_q\mathsf{F}_{0,m}\big(\emptyset\big) + 20\mathsf{F}_{0,m}\big(\tau_0(W)\big)\\
&= 20\mathsf{F}_{0,m}\big(\tau_0(W)\big)\,.
\end{aligned}$$

Example 4.2. The above example in genus 0 illustrates how the second last term in the holomorphic anomaly equation (3.2) plays a role. We consider

$$\mathsf{F}_{1,m}\big(\tau_1(W)\tau_0(W)\big)$$

to show how the last term, i.e. the term involving σ, interacts non-trivially with the other terms. The corresponding series $\mathsf{H}_{1,m}$ are

$$\begin{aligned}\mathsf{H}_{1,m}\big(\tau_1(W)\tau_0(W)\big) &= 2\mathsf{F}_{0,m}\big(\tau_1(W)\tau_0(W)\tau_0(\mathsf{1})\tau_0(F)\big)\\ &\quad - 2\Big(\mathsf{F}_{1,m}\big(\tau_2(\mathsf{1})\tau_0(W)\big) + \mathsf{F}_{1,m}\big(\tau_1(W)\tau_1(\mathsf{1})\big)\Big)\\ &\quad + \frac{20}{m}\Big(\mathsf{F}_{1,m}\big(\tau_1(F)\tau_0(W)\big) + \mathsf{F}_{1,m}\big(\tau_1(W)\tau_0(F)\big)\Big)\\ &\quad - \frac{2}{m}\mathsf{F}_{1,m}\big(\psi_1;\Delta_{U^\perp}\big)\,,\\ \mathsf{H}_{1,m}\big(\tau_1(W)\big) &= 2\mathsf{F}_{0,m}\big(\tau_1(W)\tau_0(\mathsf{1})\tau_0(F)\big)\\ &\quad - 2\mathsf{F}_{1,m}\big(\tau_2(\mathsf{1})\big)\\ &\quad + \frac{20}{m}\mathsf{F}_{1,m}\big(\tau_1(F)\big)\,.\end{aligned}$$

Let $k = -8$ be the weight of $\mathsf{F}_{1,m}\big(\tau_1(W)\big)$. Then (3.8) is equivalent to

$$\mathsf{H}_{1,m}\big(\tau_1(W)\tau_0(W)\big) = \mathsf{D}_q\mathsf{H}_{1,m}\big(\tau_1(W)\big) - 2k\mathsf{F}_{1,m}\big(\tau_1(W)\big)\,.$$

The term $\mathsf{F}_{1,m}\big(\psi_1;\Delta_{U^\perp}\big)$ can be computed using

$$\psi_1 = [\delta_1] + \frac{1}{24}[\delta_0] \in A^1(\overline{M}_{1,2})\,,$$

where $[\delta_0] \in A^1(\overline{M}_{1,2})$ is the class of the pushforward of the fundamental class under the map

$$\overline{M}_{0,4} \to \overline{M}_{1,2}$$

gluing the third and fourth markings and $[\delta_1]$ is the class of the boundary divisor of curves with a rational component carrying both markings. The genus 0 contribution vanishes by the divisor equation. Since the rank of $U^\perp$ is 20, we obtain the genus 1 contribution

$$\mathsf{F}_{1,m}\big(\psi_1;\Delta_{U^\perp}\big) = 20\mathsf{F}_{1,m}\big(\tau_0(\mathsf{p})\big)\,.$$

The divisor equation for F implies that

$$\frac{20}{m}\mathsf{F}_{1,m}\big(\tau_1(W)\tau_0(F)\big) = 20\mathsf{F}_{1,m}\big(\tau_1(W)\big) + \frac{20}{m}\mathsf{F}_{1,m}\big(\tau_0(\mathsf{p})\big)\,.$$

We can now verify the compatibility by a direct computation using divisor and dilaton equation:

$$\begin{aligned}
\mathsf{H}_{1,m}\big(\tau_1(W)\tau_0(W)\big) &= \mathsf{D}_q\mathsf{H}_{1,m}\big(\tau_1(W)\big) - 2\mathsf{F}_{1,m}\big(\tau_1(W)\big) - 2\mathsf{F}_{1,m}\big(\tau_1(W)\tau_1(\mathsf{1})\big) \\
&\quad + \frac{20}{m}\mathsf{F}_{1,m}\big(\tau_0(\mathsf{p})\big) + \frac{20}{m}\mathsf{F}_{1,m}\big(\tau_1(W)\tau_0(F)\big) \\
&\quad - \frac{2}{m}\mathsf{F}_{1,m}\big(\psi_1;\Delta_{U^\perp}\big) \\
&= \mathsf{D}_q\mathsf{H}_{1,m}\big(\tau_1(W)\big) - 4\mathsf{F}_{1,m}\big(\tau_1(W)\big) \\
&\quad + \frac{20}{m}\mathsf{F}_{1,m}\big(\tau_0(\mathsf{p})\big) + \frac{20}{m}\mathsf{F}_{1,m}\big(\tau_1(W)\tau_0(F)\big) \\
&\quad - \frac{40}{m}\mathsf{F}_{1,m}\big(\tau_0(\mathsf{p})\big) \\
&= \mathsf{D}_q\mathsf{H}_{1,m}\big(\tau_1(W)\big) + 16\mathsf{F}_{1,m}\big(\tau_1(W)\big)\,.
\end{aligned}$$

5 Relative holomorphic anomaly equation

In this section, we first state the degeneration formula for the reduced virtual class under the degeneration to the normal cone. For primitive curve class, the formula is proven in [29]. For sake of completeness, we summarize a proof for arbitrary divisibility in Appendix 8. Then, we state the relative holomorphic anomaly equation and prove the compatibility with the degeneration formula.

5.1 Degeneration formula

Let $S \to \mathbb{P}^1$ be an elliptic K3 surface with a section. For $m \geq 1$, let $\beta = mB + hF$ be a curve class. Choose a smooth fiber E of $S \to \mathbb{P}^1$. Let $\epsilon\colon \mathcal{S} \to \mathbb{A}^1$ be the total space of the degeneration to the normal cone of E in S. This space corresponds to the degeneration

$$S \rightsquigarrow S \cup_E \mathbb{P}^1 \times E\,. \tag{3.9}$$

Over the center $\iota : 0 \hookrightarrow \mathbb{A}^1$, the fiber is $S \cup_E \mathbb{P}^1 \times E$ and over $t \neq 0$, the fiber is isomorphic to S. Let $\overline{M}_{g,n}(\epsilon,\beta)$ be the moduli space of stable maps to the degeneration $\mathcal{S}$. Over $t \neq 0$, this moduli space is isomorphic to $\overline{M}_{g,n}(S,\beta)$ and over $t = 0$, this moduli space parametrizes stable maps to the expanded target

$$\widetilde{\mathcal{S}}_0 = S \cup_E \mathbb{P}^1 \times E \cup_E \cdots \cup_E \mathbb{P}^1 \times E\,.$$

Let

$$\nu = (g_1, g_2, n_1, n_2, h_1, h_2)$$

be a splitting of the discrete data g, n, h and let $\beta_i = mB + h_iF$ be the splitting of the curve class. An ordered partition of m

$$\mu = (\mu_1, \dots, \mu_l)$$

specifies the contact order along the relative divisor E.

Let $l = \text{length}(\mu)$ and $\overline{M}_{g,n}(\mathcal{S}_0, \nu)_\mu$ be the fiber product

$$\overline{M}_{g,n}(\mathcal{S}_0, \nu)_\mu = \overline{M}_{g_1,n_1}(S/E, \beta_1)_\mu \times_{E^l} \overline{M}^{\bullet}_{g_2,n_2}(\mathbb{P}^1 \times E/E, \beta_2)_\mu \tag{3.10}$$

of the boundary evaluations at relative markings[13] and let

$$\iota_{\nu\mu} \colon \overline{M}_{g,n}(\mathcal{S}_0, \nu)_\mu \to \overline{M}_{g,n}(\mathcal{S}_0, \beta)$$

be the finite morphism. Let $\Delta_{E^l} \colon E^l \to E^l \times E^l$ be the diagonal embedding.

Theorem 5.1. The reduced virtual class of maps to the degeneration (3.9) satisfies the following properties.

(i) For $\iota_t \colon \{t\} \hookrightarrow \mathbb{A}^1$, the Gysin pullback of reduced class is given by

$$\iota_t^![\overline{M}_{g,n}(\epsilon, \beta)]^{red} = [\overline{M}_{g,n}(\mathcal{S}_t, \beta)]^{red}.$$

(ii) For the special fiber,

$$[\overline{M}_{g,n}(\mathcal{S}_0, \beta)]^{red} = \sum_{\nu,\mu} \frac{\prod_i \mu_i}{l!} \iota_{\nu\mu*}[\overline{M}_{g,n}(\mathcal{S}_0, \nu)_\mu]^{red}.$$

(iii) On the special fiber, we have the factorization

$$[\overline{M}_{g,n}(\mathcal{S}_0, \nu)_\mu]^{red} = \Delta_{E^l}^! \Big([\overline{M}_{g_1,n_1}(S/E, \beta_1)_\mu]^{red} \times [\overline{M}^{\bullet}_{g_2,n_2}(\mathbb{P}^1 \times E/E, \beta_2)_\mu]^{vir} \Big).$$

Proof. When $m \geq 1$, the reduced class of the disconnected moduli space $\overline{M}^{\bullet}_{g,n}(S/E, \beta)$ vanishes on all components parameterizing maps with at least two connected components. Therefore, disconnected theory can only appear on the bubble $\mathbb{P}^1 \times E$. The proof is given in Appendix 8. □

[13]We put $\bullet$ to indicate (possibly) disconnected theory. Namely, for each connected component C of the domain curve, intersection of C with the relative divisor E is nontrivial.

Denote an ordered cohomology weighted partition by

$$\underline{\mu} = \big((\mu_1, \delta_1), \ldots, (\mu_l, \delta_l)\big)\,,\ \delta_i \in H^*(E)$$

and let $\omega \in H^2(E)$ be the point class. The descendent potential for the pair (S, E) is defined analogously to the absolute case:

$$\mathsf{F}^{\mathrm{rel}}_{g,m}\big(\alpha; \gamma_1, \ldots, \gamma_n \mid \underline{\mu}\big) = \sum_{h \geq 0} \big\langle \alpha; \gamma_1, \ldots, \gamma_n \mid \underline{\mu} \big\rangle^{S/E}_{g, mB+hF} q^{h-m}\,.$$

The descendent potential for the pair $(\mathbb{P}^1 \times E, E)$ is defined by

$$\mathsf{G}^{\mathrm{rel},\bullet}_{g,m}\big(\alpha; \gamma_1, \ldots, \gamma_n \mid \underline{\mu}\big) = \sum_{h \geq 0} \big\langle \alpha; \gamma_1, \ldots, \gamma_n \mid \underline{\mu} \big\rangle^{\mathbb{P}^1 \times E/E, \bullet}_{g, mB+hF} q^{h}\,.$$

As a corollary, we get the degeneration formula of reduced Gromov–Witten invariants.

Corollary 5.2. Let $\gamma_1, \ldots, \gamma_n \in H^*(S)$ and choose a lift of these cohomology classes to the total space $\mathcal{S}$. Then

$$\mathsf{F}_{g,m}\big(\tau_{a_1}(\gamma_1) \ldots \tau_{a_n}(\gamma_n)\big) = \sum_{\nu} \sum_{\underline{\mu} \neq \underline{\mu_\omega}} \frac{\prod_i \mu_i}{l!} \mathsf{F}^{\mathrm{rel}}_{g_1,m}\big(\ldots \mid \underline{\mu}\big) \cdot \mathsf{G}^{\mathrm{rel},\bullet}_{g_2,m}\big(\ldots \mid \underline{\mu}^\vee\big)\,, \quad (3.11)$$

where

$$\underline{\mu}^\vee = \big((\mu_1, \delta_1^\vee), \ldots, (\mu_l, \delta_l^\vee)\big) \text{ and } \underline{\mu_\omega} = \big((\mu_1, \omega), \ldots, (\mu_l, \omega)\big)\,.$$

Proof. By Theorem 5.1, we are left to prove that the relative profile $\underline{\mu_\omega}$ on S/E has vanishing contribution. Let x be the intersection of the section of the elliptic fibration and the fiber E. We consider (E, x) as an abelian variety. Let K be the kernel of the following morphism between abelian varieties

$$E^l \to \mathsf{Pic}^0(E)\,, (x_i)_i \mapsto \mathcal{O}_E\Big(\sum_i \mu_i(x_i - x)\Big)\,.$$

Consider a stable map f from a curve C to an expanded degeneration of S/E. The equality $f_*[C] = \beta_1$ (after pushforward to S) in $H_2(S, \mathbb{Z})$ lifts to a rational equivalence of line bundles on S because the cycle-class map

$$c_1 \colon \mathsf{Pic}(S) \to H^2(S, \mathbb{Z}) \cong H_2(S, \mathbb{Z})$$

is injective. Intersecting with the relative divisor, the two line bundles are, respectively, $\mathcal{O}_E(\sum \mu_i x_i)$ and $\mathcal{O}_E(mx)$. Thus, we see that the evaluation map $\overline{M}_{g_1,n_1}(S/E, \beta_1) \to E^l$ factors through K. Since $K \subset E^l$ has codimension 1 a generic point on E^l does not lie on K and thus the contribution from the relative profile $\underline{\mu_\omega}$ vanishes. $\square$

5.2 Relative holomorphic anomaly equations

Assuming quasimodularity, we have two ways to compute the derivative of $\mathsf{F}_{g,m}$ with respect to C_2:

(i) Apply the degeneration formula Corollary 5.2, together with the holomorphic anomaly equations for (S, E) and $(\mathbb{P}^1 \times E, E)$.

(ii) Apply the holomorphic anomaly equation (3.3) for S, followed by the degeneration formula for each term.

We argue that both ways yield the same result. This compatibility is parallel to the compatibility proved in [34, Section 4.6]. We first state the holomorphic anomaly equations for the relevant relative geometries.

Relative $(\mathbb{P}^1 \times E, E)$

Consider $\pi\colon \mathbb{P}^1 \times E \to \mathbb{P}^1$ as a trivial elliptic fibration over $\mathbb{P}^1$. For the pair $(\mathbb{P}^1 \times E, E)$ the holomorphic anomaly equation holds for cycle-valued generating series [34]. The equation for descendent potentials can thus be obtained by integrating against tautological classes $\alpha \in R^*(\overline{M}_{g,n})$. For insertions $\gamma_i \in H^*(\mathbb{P}^1 \times E, \mathbb{Q})$ we will simply write γ. Let $\underline{\mu} = \big((\mu_1, \delta_1), \dots, (\mu_l, \delta_l)\big)$ and $\underline{\mu}'$ be ordered cohomology weighted partitions. We denote by

$$\mathsf{G}^{\sim,\bullet}_{g,m}(\underline{\mu} \mid \alpha; \gamma \mid \underline{\mu}') = \sum_{h \geq 0} \big\langle \underline{\mu} \mid \alpha; \gamma \mid \underline{\mu}' \big\rangle^{\mathbb{P}^1 \times E, \sim, \bullet}_{g, m\mathbb{P}^1 + hE} q^h$$

the disconnected rubber generating series for $\mathbb{P}^1 \times E$ relative to divisors at 0 and ∞. Let $\Delta_E \subset E \times E$ be the class of the diagonal. Define the generating series

$$\begin{aligned}
&\mathsf{P}^{\mathrm{rel},\bullet}_{g,m}(\alpha; \gamma \mid \underline{\mu}) \\
&= \mathsf{G}^{\mathrm{rel},\bullet}_{g-1,m}(\alpha; \gamma, \Delta_{\mathbb{P}^1} \mid \underline{\mu}) \\
&\quad + 2 \sum_{\substack{g = g_1 + g_2 \\ \{1,\dots,n\} = I_1 \sqcup I_2 \\ \forall i \in I_2 : \gamma_i \in H^2(E) \\ h \geq 0}} \sum_{\substack{b; b_1, \dots, b_h \\ l_1, \dots, l_h}} \frac{\prod_{i=1}^h b_i}{h!} \mathsf{G}^{\mathrm{rel},\bullet}_{g_1,m}\big(\alpha_{I_1}; \gamma_{I_1} \mid ((b,1), (b_i, \Delta_{E,\ell_i})_{i=1}^h)\big) \\
&\qquad\qquad\qquad \times \mathsf{G}^{\sim,\bullet}_{g_2,m}\big(((b,1),(b_i, \Delta^\vee_{E,\ell_i})_{i=1}^h) \mid \alpha_{I_2}; \gamma_{I_2} \mid \underline{\mu}\big) \\
&\quad - 2 \sum_{i=1}^n \mathsf{G}^{\mathrm{rel},\bullet}_{g,m}(\alpha\psi_i; \gamma_1, \dots, \gamma_{i-1}, \pi^*\pi_*\gamma_i, \gamma_{i+1}, \dots, \gamma_n \mid \underline{\mu}) \\
&\quad - 2 \sum_{i=1}^l \mathsf{G}^{\mathrm{rel},\bullet}_{g,m}\big(\alpha; \gamma \mid (\mu_1, \delta_1), \dots, (\mu_i, \psi_i^{\mathrm{rel}} \pi^* \pi_* \delta_i), \dots, (\mu_l, \delta_l)\big)
\end{aligned}$$

where ψ_i^{rel} is the cotangent line class at the i-th relative marking and $\Delta_E = \sum \Delta_{E,l_i} \otimes \Delta_{E,l_i}^\vee$ is the pullback of the Künneth decomposition of Δ_E at the corresponding relative marking. The holomorphic anomaly equation takes the form:

Proposition 5.3. ([34, Proposition 20]) $\mathsf{G}_{g,m}^{\mathrm{rel},\bullet}(\alpha;\gamma \mid \underline{\mu})$ is a quasimodular form and

$$\frac{d}{dC_2}\mathsf{G}_{g,m}^{\mathrm{rel},\bullet}(\alpha;\gamma \mid \underline{\mu}) = \mathsf{P}_{g,m}^{\mathrm{rel},\bullet}(\alpha;\gamma \mid \underline{\mu})\,.$$

Relative (S,E)

Since the log canonical bundle of (S,E) is nontrivial, relative moduli spaces in fiber direction have nontrivial virtual fundamental class. Define

$$\mathsf{F}_{g,0}^{vir-\mathrm{rel}}(\alpha;\gamma \mid \emptyset) = \sum_{h\geq 0} \big\langle \alpha;\gamma \mid \emptyset \big\rangle_{g,hF}^{S/E,vir} q^h\,.$$

Recall that we denote the pullback of the diagonal of $\mathbb{P}^1$ as

$$\Delta_{\mathbb{P}^1} = 1 \boxtimes F + F \boxtimes 1 = \sum_{i=1}^{2} \delta_i \boxtimes \delta_i^\vee\,.$$

Define a generating series

$$
\begin{aligned}
&\mathsf{H}^{\mathrm{rel}}_{g,m}\big(\alpha;\gamma \mid \underline{\mu}\big) \\
&= \mathsf{F}^{\mathrm{rel}}_{g-1,m}\big(\alpha;\gamma,\Delta_{\mathbb{P}^1} \mid \underline{\mu}\big) \\
&\quad + 2 \sum_{\substack{g=g_1+g_2 \\ \{1,\dots,n\}=I_1\sqcup I_2 \\ i\in\{1,2\}}} \mathsf{F}^{\mathrm{rel}}_{g_1,m}\big(\alpha_{I_1};\gamma_{I_1},\delta_i \mid \underline{\mu}\big)\, \mathsf{F}^{vir-\mathrm{rel}}_{g_2,0}\big(\alpha_{I_2};\gamma_{I_2},\delta_i^\vee \mid \emptyset\big) \\
&\quad + 2 \sum_{\substack{g=g_1+g_2 \\ \{1,\dots,n\}=I_1\sqcup I_2 \\ \forall i\in I_2:\gamma_i\in H^2(E) \\ h\geq 0}} \sum_{\substack{b;b_1,\dots,b_h \\ l_1,\dots,l_h}} \frac{\prod_{i=1}^h b_i}{h!} \mathsf{F}^{\mathrm{rel}}_{g_1,m}\big(\alpha_{I_1};\gamma_{I_1} \mid ((b,1),(b_i,\Delta_{E,\ell_i})_{i=1}^h)\big) \\
&\qquad\qquad\qquad\qquad \times \mathsf{G}^{\sim,\bullet}_{g_2,m}\big(((b,1),(b_i,\Delta^\vee_{E,\ell_i})_{i=1}^h) \mid \alpha_{I_2};\gamma_{I_2} \mid \underline{\mu}\big) \\
&\quad - 2\sum_{i=1}^n \mathsf{F}^{\mathrm{rel}}_{g,m}\big(\alpha\psi_i;\gamma_1,\dots,\gamma_{i-1},\pi^*\pi_*\gamma_i,\gamma_{i+1},\dots,\gamma_n \mid \underline{\mu}\big) \\
&\quad - 2\sum_{i=1}^l \mathsf{F}^{\mathrm{rel}}_{g,m}\big(\alpha;\gamma \mid ((\mu_1,\delta_1),\dots,(\mu_i,\psi_i^{\mathrm{rel}}\pi^*\pi_*\delta_i),\dots,(\mu_l,\delta_l))\big) \\
&\quad + \frac{20}{m}\sum_{i=1}^n \langle \gamma_i, F\rangle \mathsf{F}^{\mathrm{rel}}_{g,m}\big(\alpha;\gamma_1,\dots,\gamma_{i-1},F,\gamma_{i+1},\dots,\gamma_n \mid \underline{\mu}\big) \\
&\quad - \frac{2}{m}\sum_{i<j} \mathsf{F}^{\mathrm{rel}}_{g,m}\big(\alpha;\gamma_1,\dots,\underbrace{\sigma_1(\gamma_i,\gamma_j)}_{i\text{th}},\dots,\underbrace{\sigma_2(\gamma_i,\gamma_j)}_{j\text{th}},\dots,\gamma_n \mid \underline{\mu}\big)\,.
\end{aligned}
$$

The conjectural holomorphic anomaly equation for (S,E) has the following form:

$$
\mathsf{F}^{\mathrm{rel}}_{g,m}(\alpha;\gamma \mid \underline{\mu}) \in \frac{1}{\Delta(q)^m}\mathsf{QMod}(m)
$$

and

$$
\frac{d}{dC_2}\mathsf{F}^{\mathrm{rel}}_{g,m}(\alpha;\gamma \mid \underline{\mu}) = \mathsf{H}^{\mathrm{rel}}_{g,m}(\alpha;\gamma \mid \underline{\mu})\,. \tag{3.12}
$$

Proposition 5.4. Let $m \geq 1$. Assuming quasimodularity for $\mathsf{F}_{g,m}$ and $\mathsf{F}^{\mathrm{rel}}_{g,m}$, the holomorphic anomaly equations are compatible with the degeneration formula in the above sense.

Proof. The proof given in [34, Proposition 21] treats virtual fundamental classes, not reduced classes. The splitting behavior of the reduced class with respect to restriction to boundary divisors [29, Section 7.3] calls for a slight adaptation of the proof. For this, we introduce a formal variable ε with $\varepsilon^2 = 0$. We can then interpret reduced Gromov–Witten invariants of the K3 surface as integrals against the class[14]

$$[\overline{M}_{g,n}(S,\beta)]^{vir} + \varepsilon\, [\overline{M}_{g,n}(S,\beta)]^{red}$$

followed by taking the $[\varepsilon]$-coefficient. We consider a similar class for S/E. This class has the advantage of satisfying the usual splitting behavior of virtual fundamental classes. Thus, for this class one can follow the proof of compatibility given in [34, Proposition 21]. All the terms appearing in the computation (ii) also appear in computation (i). We are left with proving the cancellation of the remaining terms in (i). This follows from comparing ψ_i^{rel}-class and the ψ-class pulled-back from the stack of target degeneration [34, Lemma 22]. In particular, we match the following terms: the third term of $\mathsf{H}^{\mathrm{rel}}$ times $\mathsf{G}^{\mathrm{rel},\bullet}$ with the fourth term of $\mathsf{F}^{\mathrm{rel}}$ times $\mathsf{P}^{\mathrm{rel},\bullet}$; and analogously for the fifth term of $\mathsf{H}^{\mathrm{rel}}$ times $\mathsf{G}^{\mathrm{rel},\bullet}$ with the second term of $\mathsf{F}^{\mathrm{rel}}$ times $\mathsf{P}^{\mathrm{rel},\bullet}$. □

The main advantage of the holomorphic anomaly equation is that it is compatible with the degeneration formula. Thus, the genus reduction from the degeneration formula connects the low genus results with arbitrary genus predictions. On the other hand, it is not even clear to say what should be the compatibility of the multiple cover formula and the degeneration formula.

6 Tautological relations and initial condition

This section contains a proof of the multiple cover formula in genus 0 and genus 1 for any divisibility m. It is a direct consequence of the KKV formula. However, as initial condition for our induction we also require a special case in genus 2, which cannot be easily deduced from the KKV formula. We treat this descendent potential separately, using double ramification relations [3] for K3 surfaces. This approach is likely to give *relations in any genus* and will be pursued in the future.

6.1 Double ramification relations

In this section we recall *double ramification relations with target variety* developed in [2, 3].

Let $\mathfrak{Pic}_{g,n}$ be the Picard stack for the universal curve over the stack of prestable curves $\mathfrak{M}_{g,n}$ of genus g with n markings. Let

$$\pi\colon \mathfrak{C} \to \mathfrak{Pic}_{g,n},\ s_i\colon \mathfrak{Pic}_{g,n} \to \mathfrak{C},\ \mathfrak{L} \to \mathfrak{C},\ \omega_\pi \to \mathfrak{C} \tag{3.13}$$

[14] We thank G. Oberdieck for pointing this out.

be the universal curve, the i-th section, the universal line bundle and the relative dualizing sheaf of π. The following operational Chow classes on $\mathfrak{Pic}_{g,n}$ are obtained from the universal structure (3.13):

- $\psi_i = c_1(s_i^*\omega_\pi) \in A^1_{\mathsf{op}}(\mathfrak{Pic}_{g,n})$,
- $\xi_i = c_1(s_i^*\mathfrak{L}) \in A^1_{\mathsf{op}}(\mathfrak{Pic}_{g,n})$,
- $\eta = \pi_*\left(c_1(\mathfrak{L})^2\right) \in A^1_{\mathsf{op}}(\mathfrak{Pic}_{g,n})$.

Let $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ be a vector of integers satisfying

$$\sum_i a_i = d\,, \tag{3.14}$$

where d is the degree of the line bundle. We denote by $P^{c,r}_{g,A,d}$ the codimension c component of the class

$$\sum_{\substack{\Gamma\in\mathsf{G}_{g,n,d}\\ w\in\mathsf{W}_{\Gamma,r}}} \frac{r^{-h^1(\Gamma_\delta)}}{|\operatorname{Aut}(\Gamma_\delta)|}\, j_{\Gamma*}\left[\prod_{i=1}^n \exp\left(\frac{1}{2}a_i^2\psi_i + a_i\xi_i\right)\prod_{v\in V(\Gamma_\delta)}\exp\left(-\frac{1}{2}\eta(v)\right)\right.$$
$$\left.\prod_{e=(h,h')\in E(\Gamma)} \frac{1-\exp\left(-\frac{w(h)w(h')}{2}(\psi_h+\psi_{h'})\right)}{\psi_h+\psi_{h'}}\right].$$

We refer to [3] for details about the notations. This expression is polynomial in r when r is sufficiently large. Let $P^c_{g,A,d}$ be the constant part of $P^{c,r}_{g,A,d}$.

Theorem 6.1. ([3, Theorem 8]) $P^c_{g,A,d} = 0$ for all $c > g$ in $A^c_{\mathsf{op}}(\mathfrak{Pic}_{g,n})$.

After restricting $P^c_{g,A,d}$ to (3.14), this expression is a polynomial in $a_1, \ldots, a_{n-1}$. The polynomiality will be used to get refined relations.

Let L be a line bundle on S with degree

$$\int_\beta c_1(L) = d\,.$$

The choice of a line bundle L induces a morphism

$$\varphi_L \colon \overline{M}_{g,n}(S,\beta) \to \mathfrak{Pic}_{g,n},\ [f\colon C\to S] \mapsto (C, f^*L)\,.$$

Then Theorem 6.1 gives relations

$$P^c_{g,A,d}(L) = \varphi_L^* P^c_{g,A,d} \cap [\overline{M}_{g,n}(S,\beta)]^{red} = 0 \text{ for all } c > g \tag{3.15}$$

in $A_{g+n-c}\big(\overline{M}_{g,n}(S,\beta)\big)$.

6.2 Compatibility II

The relations among descendent potentials coming from tautological relations on $\overline{M}_{g,n}(S,\beta)$ are compatible with the multiple cover formula. This follows from two observations. Firstly, the splitting behavior of the reduced class, discussed in Section 3.2, is crucial. It is already crucial to justify compatibility with respect to boundary restriction for tautological classes pulled back from $\overline{M}_{g,n}$. For tautological relations on $\overline{M}_{g,n}(S,\beta)$, a second fact, which we want to explain below, is essential for the compatibility.

For $c > g > 0$, $A \in \mathbb{Z}^n$ and $b \in \mathbb{Z}$, consider the series of relations

$$P^c_{g,bA,db}(L^{\otimes b}) = 0$$

obtained by tensoring the line bundle L by b times. For each coefficient of a monomial in a_i-variables, this expression is polynomial in b and hence each of b-variable is a relation. As a consequence, each term of a relation $P^c_{g,A,m}(F)$ gives the same value of

$$m^{\deg - \underline{\deg}},$$

where $\deg(\xi) = 1$ and $\underline{\deg}(\xi) = 0$, as in Definition 1.1. The same holds true with the roles of F and W interchanged. Thus, the relations are compatible with the operator

$$m^{\deg - \underline{\deg}} \mathsf{T}_{m,2g-2+\underline{\deg}},$$

which gives the multiple cover formula in Conjecture 3.1.

6.3 Initial condition

The Katz–Klemm–Vafa (KKV) formula implies that the generating series of λ_g-integrals

$$\mathsf{F}_{g,m}(\lambda_g; \emptyset)$$

satisfy the multiple cover formula [36]. Here, $\lambda_g = c_g(\mathbb{E}_g)$ is the top Chern class of the rank g Hodge bundle $\mathbb{E}_g$ on $\overline{M}_g(S,\beta)$. The KKV formula will be the starting point of our genus induction.

The class λ_g is a tautological class by the Grothendieck–Riemann–Roch computation ([15]) but the formula is rather complicated. Instead we use an alternative expression of λ_g in terms of double ramification cycle, proven in [17]. We recall that the class $(-1)^g\lambda_g$ is equal to the double ramification cycle $\mathsf{DR}_g(\emptyset)$ with the empty condition. By [17, Theorem 1] the class $\mathsf{DR}_g(\emptyset)$ can be written as a graph sum of tautological classes *without* κ-classes.

Proposition 6.2. The multiple cover formula holds in genus 0 and genus 1 for all $m \geq 1$.

Proof. When $g = 0, 1$, the tautological ring $R^*(\overline{M}_{g,n})$ is additively generated by boundary strata ([19, 37]). Thus, one can replace descendents $\alpha \in R^*(\overline{M}_{g,n})$ by classes in $H^*(S)$. By the divisor equation and the dimension constraint, we can reduce to the case $\mathsf{F}_{0,m}(\emptyset)$ and $\mathsf{F}_{1,m}(\tau_0(\mathsf{p}))$. The genus 0 case is covered by the full Yau–Zaslow formula [21, 36]. The genus 1 case follows from the genus 2 KKV formula. Using the boundary expression of λ_2 on $\overline{M}_2$, we have

$$\begin{aligned}\mathsf{F}_{2,m}\big(\lambda_2;\emptyset\big) &= \frac{1}{240}\mathsf{F}_{1,m}\big(\psi_1;\Delta_S\big) + \frac{1}{1152}\mathsf{F}_{0,m}\big(\,;\Delta_S,\Delta_S\big)\\ &= \frac{1}{10}\mathsf{F}_{1,m}\big(\tau_0(\mathsf{p})\big) + \frac{1}{60}\mathsf{D}_q^2\mathsf{F}_{0,m}\big(\emptyset\big)\,,\end{aligned}$$

where $\Delta_S \subset S \times S$ is the diagonal class. Therefore, $\mathsf{F}_{1,m}\big(\tau_0(\mathsf{p})\big)$ satisfies Conjecture 3.1. □

In the argument below, we will use tautological relations on $\overline{M}_{g,n}$ which are recently obtained by r-spin relations. For convenience, we summarize the result.

Proposition 6.3. ([23]) Let $g \geq 2$ and $n \geq 1$. Consider tautological classes on $\overline{M}_{g,n}$.

(i) (Topological Recursion Relations) Any monomial of ψ-classes of degree at least g can be represented by a tautological class supported on boundary strata without κ-classes.

(ii) Any tautological class of degree $g-1$ can represented by a sum of a linear combination of $\psi_1^{g-1}, \ldots, \psi_n^{g-1}$ and a tautological class supported on boundary strata.

Proof. The proof of (i) follows from the proof of [23, Lemma 5.2] (see also [12, page 3]). By [23, Proposition 3.1] (or [10, Theorem 1.1]) the degree $g-1$ part $R^{g-1}(\mathcal{M}_{g,n})$ is spanned by $\psi_1^{g-1}, \ldots, \psi_n^{g-1}$. Since relations used in the proof are all tautological, the boundary expression is tautological and thus we obtain (ii). □

Together with the boundary expression for λ_{g+1} we obtain the following more general consequence of the KKV formula:

Proposition 6.4. Let $m \geq 1$ and $g \geq 1$. Assume the multiple cover formula Conjecture 3.1 holds for m and all descendents of genus $< g$. Then Conjecture 3.1 holds for

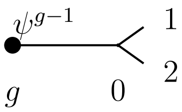
$$\mathsf{F}_{g,m}\big(\tau_{g-1}(\mathsf{p})\big)\,.$$

Proof. Let $\delta \in R^1(\overline{M}_g)$ be the boundary divisor corresponding to a curve with nonseparating node. Denote two half edges as h and h'. Recall that $(-1)^g\lambda_g$ is equal to the double ramification cycle $\mathsf{DR}_g(\emptyset)$ with the empty condition. We use this formula for genus $g+1$. By [17, Theorem 1],

$$\begin{aligned}(-1)^{g+1}\lambda_{g+1} &= \mathsf{DR}_{g+1}(\emptyset)\\ &= \frac{1}{2}\Big[-\frac{1}{(g+1)!}\sum_{w=0}^{r-1}\big(\frac{w^2}{2}(\psi_h+\psi_{h'})\big)^g\Big]_{r^1}\delta + \text{ lower genus}\,,\end{aligned}$$

where $[\cdots]_{r^1}$ is the coefficient of the linear part of a polynomial in r. The leading term is nonzero by Faulhaber's formula.

By Proposition 6.3 (i) any ψ-monomial in $R^{\geq g}(\overline{M}_{g,n})$ can be represented by a sum of tautological classes supported on boundary strata without κ classes. There is only one graph with a genus g vertex (with a rational component carrying both markings). The graph is decorated with a polynomial of degree $g-1$ in ψ- and κ-classes. By Proposition 6.3 (ii) this tautological class can be represented by a sum of a multiple of ψ^{g-1} and tautological classes supported on boundary strata. We find that [15]

$(\psi_1+\psi_2)^g = c$ 

$+$ lower genus

in $R^g(\overline{M}_{g,2})$ for some $c \in \mathbb{Q}$. Therefore, it suffices to prove that c is nonzero. Recall that $\lambda_g\lambda_{g-1}$ vanishes on $\overline{M}_{g,n} \setminus M^{rt}_{g,n}$, so

$$\int_{\overline{M}_{g,2}} (\psi_1+\psi_2)^g\lambda_g\lambda_{g-1} = c\int_{\overline{M}_{g,1}} \psi_1^{g-1}\lambda_g\lambda_{g-1}\,.$$

The left hand side of the equation is nonzero by [17, Lemma 8], which concludes the proof. □

We now consider the case of genus two. By the Getzler–Ionel vanishing on $\overline{M}_{2,n}$, the dimension constraint, and the divisor equation any descendent insertion reduces to the following three cases:

$$\mathsf{F}_{2,m}\big(\tau_1(\mathsf{p})\big)\,,\quad \mathsf{F}_{2,m}\big(\tau_0(\mathsf{p})^2\big)\,,\quad \mathsf{F}_{2,m}\big(\tau_1(\gamma)\tau_0(\mathsf{p})\big)\quad \text{with } \gamma\in H^2(S)\,.$$

The first case is treated in Proposition 6.4 and follows from the KKV formula in genus three and lower genus. The second case for $m=2$ is treated as part of the proof of Theorem 1.1 in Section 7. We use the double ramification relation (3.15) to prove the multiple cover formula for the third case. The point class p will be obtained as the product of F and W.

[15]The number under each vertex is the genus and legs correspond to markings.

Proposition 6.5. For $\gamma \in H^2(S)$, the generating series $\mathsf{F}_{2,m}(\tau_1(\gamma)\tau_0(\mathsf{p}))$ satisfies Conjecture 3.1.

Proof. We will use relations in $A_{2+n-3}\left(\overline{M}_{2,n}(S,\beta)\right)$:

$$P^3_{2,A,m}(F) = 0$$

associated to the line bundle $\mathcal{O}_S(F)$ on S. More precisely, we will distinguish two cases $\gamma \in U$ and $\gamma \in U^\perp$ and set respectively

$$A = (a_1\,, m - a_1)\,, \quad A = (a_1\,, a_2\,, m - a_1 - a_2)\,.$$

Refined relations are then obtained by considering particular monomials in the a_i, as outlined in the previous section. The η-class vanishes in this case because $\langle F, F\rangle = 0$ and, for the same reason, ξ_i^2 vanishes. Define

$$X = \mathsf{F}_{2,m}\big(\tau_1(\gamma)\tau_0(\mathsf{p})\big)\,.$$

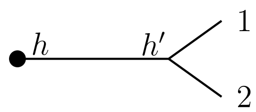
The case $\gamma = F$ is treated first. As explained in Section 6.1, the tautological relations are polynomial in a_i and we may obtain a refined relation by considering the $[a_1^4]$-coefficient of

$$P^3_{2,A,m}(F)|_{a_2 = m - a_1}\,.$$

We will only need to consider boundary strata which both:

- contribute to X and
- contribute to the $[a_1^4]$-coefficient.

These two properties simplify the calculation significantly. By the splitting property of the reduced class, a relevant boundary stratum is a tree with one genus 2 vertex and contracted genus 0 components. The integrals are given by the intersection product of the corresponding insertions. In the case with only two markings, the only relevant stratum is[16]



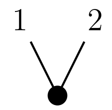
.

The weight factor for this stratum is

$$\frac{w(h)w(h')}{2} = -\frac{m^2}{2}\,.$$

[16]The genus 2 vertex is represented by a filled node and other nodes represent genus 0 vertices. Labeled half-edges correspond to markings.

This stratum, therefore, cannot contribute to the $[a_1^4]$-coefficient, since ψ-classes on the genus 0 component vanish. It remains to determine the contributions from the trivial graph



We will order the terms by the total degree $\deg(\psi)$ in the ψ-classes.

() $\deg(\psi) = 0$. The relation we consider is of codimension three. This case is therefore impossible by virtue of $\xi_i^2 = 0$.

(i) $\deg(\psi) = 1$. This case results in non-trivial terms, discussed below.

(ii) $\deg(\psi) \geq 2$. We may apply Proposition 6.3 (i) to reduce to the descendent $\mathsf{F}_{2,m}\big(\tau_1(\mathsf{p})\big)$. This descendent is covered by Proposition 6.4.

Therefore, up to lower genus data, the $[a_1^4]$-coefficient is

$$-\frac{1}{2}\psi_1\xi_1\xi_2 - \frac{1}{2}\psi_2\xi_1\xi_2\,.$$

Integrating

$$\mathrm{ev}_2^*(W)P^3_{2,A,m}(F)|_{a_2=m-a_1}$$

against the reduced class, we find (up to lower genus data)

$$-\frac{1}{2}X - \frac{m}{2}\mathsf{F}_{2,m}\big(\tau_1(\mathsf{p})\big)\,,$$

where the second term is obtained by application of the divisor equation. We thus find that X is a linear combination of terms which satisfy Conjecture 3.1. Switching the role of F and W, we obtain the same result for $\gamma = W$.

Next, we consider $\gamma \in U^\perp$. The following vanishing of intersection products will be used frequently:

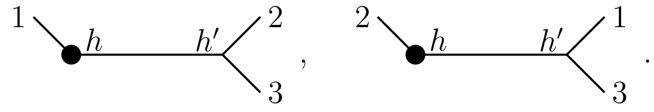
$$\langle\gamma, F\rangle = 0\,,\quad \langle\gamma, W\rangle = 0\,,\quad \langle\gamma, \beta\rangle = 0\,.$$

We use a similar argument as above, this time, however, we use three markings and consider the $[a_1^3a_2]$-coefficient of[17]

$$\mathrm{ev}_1^*(\gamma)\,\mathrm{ev}_2^*(W)P^3_{2,A,m}(F)|_{a_3=m-a_1-a_2}\,. \tag{3.16}$$

[17] We are grateful to the referee for pointing out a mistake in an earlier version of the text. It has become clear that the choice of monomial, leading to non-trivial relations, is a very subtle one. Symmetry in the a_i and the insertions causes cancellation in many cases. We plan to come back to this in future work.

By the above vanishing of intersection products, the only possible trees with non-trivial contribution are



The weight factor for the right stratum is

$$\frac{w(h)w(h')}{2} = -\frac{(m-a_2)^2}{2} .$$

Since ψ-classes on the genus 0 component vanish, the power of a_1 in any monomial obtained from this stratum is bounded by two. The contribution to the $[a_1^3 a_2]$-coefficient is, therefore, zero.

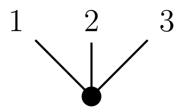
Next, we explain the contributions from the left stratum. Note that the left vertex is of genus 2 with two markings and we may apply the same reasoning as in the discussion for $\gamma = F$ above. Here, the $\deg(\psi) = 0$ term $\xi_1\xi_2$ has trivial contribution due to $\langle \gamma, F \rangle = 0$. The $\deg(\psi) = 1$ terms $\psi_h\xi_2$, $\psi_h\xi_3$ have vanishing contribution by application of the divisor equation for γ. Non-trivial contributions are obtained only from

$$\psi_1\xi_2 , \quad \psi_1\xi_3 .$$

These two terms have contributions

$$-\frac{(m-a_1)^2}{4} a_1^2 a_2 X , \quad -\frac{(m-a_1)^2}{4} a_1^2 a_3 X .$$

The $[a_1^3 a_2]$-coefficients, however, cancel due to $a_3 = m - a_1 - a_2$. It remains to determine the contributions from the trivial graph:



As above, we order the terms by the total degree $\deg(\psi)$ in the ψ-classes.

() $\deg(\psi) = 0$. The relation we consider is of codimension three. Since $\xi_i^2 = 0$, the class ξ_1 must appear. This term, however, vanishes due to $\langle \gamma, F \rangle = 0$.

(i) $\deg(\psi) = 1$. This case results in non-trivial terms corresponding to ψ_1 or ψ_3, discussed below. The choice of the monomial $[a_1^3 a_2]$ excludes the appearance of ψ_2.

(ii) $\deg(\psi) = 2$. This case results in non-trivial terms corresponding to $\psi_1\psi_3$ or ψ_3^2, discussed below. The choice of the monomial $[a_1^3 a_2]$ excludes the appearance of ψ_1^2.

(iii) $\deg(\psi) = 3$. As above, this case reduces to the descendent $\mathsf{F}_{2,m}\big(\tau_1(\mathsf{p})\big)$ which is covered already.

The contributions from $\deg(\psi) \in \{1, 2\}$ are:

$$
\begin{aligned}
\psi_1\xi_2\xi_3 \quad &\to \quad \frac{1}{2}a_1^2 a_2 a_3 \mathsf{F}_{2,m}\big(\tau_1(\gamma)\tau_0(\mathsf{p})\tau_0(F)\big) \\
&= \frac{1}{2}a_1^2 a_2 (m - a_1 - a_2) m X \,, \\
\psi_3\xi_2\xi_3 \quad &\to \quad \frac{1}{2}a_3^2 a_2 a_3 \mathsf{F}_{2,m}\big(\tau_0(\gamma)\tau_0(\mathsf{p})\tau_1(F)\big) \\
&= 0 \,, \\
\psi_1\psi_3\xi_2 \quad &\to \quad \frac{1}{2}a_1^2 \frac{1}{2}a_3^2 a_2 \mathsf{F}_{2,m}\big(\tau_1(\gamma)\tau_0(\mathsf{p})\tau_1(1)\big) \\
&= a_1^2 a_2 (m - a_1 - a_2)^2 X \,, \\
\psi_1\psi_3\xi_3 \quad &\to \quad \frac{1}{2}a_1^2 \frac{1}{2}a_3^3 \mathsf{F}_{2,m}\big(\tau_1(\gamma)\tau_0(W)\tau_1(F)\big) \\
&= \frac{1}{4}a_1^2 (m - a_1 - a_2)^3 X + \text{ (lower genus)} \,, \\
\psi_3^2\xi_2 \quad &\to \quad \frac{1}{8}a_3^4 a_2 \mathsf{F}_{2,m}\big(\tau_0(\gamma)\tau_0(\mathsf{p})\tau_2(1)\big) \\
&= \frac{1}{8}a_2 (m - a_1 - a_2)^4 X \,, \\
\psi_3^2\xi_3 \quad &\to \quad \frac{1}{8}a_3^4 a_3 \mathsf{F}_{2,m}\big(\tau_0(\gamma)\tau_0(W)\tau_2(F)\big) \\
&= 0 \,.
\end{aligned}
$$

The third calculation uses the dilaton equation. All of the other calculations are obtained by application of the divisor equation. Additionally, the fourth calculation involves Proposition 6.3. The only stratum with a genus 2 vertex (i.e. with both markings on a contracted genus 0 component) has vanishing contribution due to $\langle \gamma, F \rangle = 0$ and, therefore, the relation reduces to lower genus descendents. The total contribution to $[a_1^3 a_2]$ is

$$
-\frac{1}{2}mX - 2mX + \frac{3}{2}mX - \frac{1}{2}mX = -\frac{3}{2}mX \,.
$$

We find that X is a linear combination of terms which satisfy Conjecture 3.1.

□

Remark 6.6. In fact, for $\gamma \in U^\perp$ the above generating series vanishes (and thus trivially satisfies the multiple cover formula). A proof in the primitive case is given in [9, Lemma 4].

7 Proof of Theorem 1.1 and 1.3

7.1 Proof of Theorem 1.1

The proof proceeds via induction on the pair (g, n) ordered by the lexicographic order: $(g', n') < (g, n)$ if

- $g' < g$ or
- $g' = g$ and $n' < n$.

Recall the dimension constraint of insertions:

$$g + n = \deg(\alpha) + \sum_i \deg(\gamma_i).$$

We separate the proof into several steps.
Case 0. The genus 0 case is covered by Proposition 6.2. This serves as the start for our induction.
Case 1. If all cohomology classes γ_i satisfy $\deg(\gamma_i) \leq 1$, then $\deg(\alpha) \geq g$ and by the strong form of Getzler–Ionel vanishing [15, Proposition 2] we have $\alpha = \iota_* \alpha'$ with $\alpha' \in R^*(\partial \overline{M}_{g,n})$ and $\iota \colon \partial \overline{M}_{g,n} \to \overline{M}_{g,n}$. We are thus reduced to lower (g, n).
Case 2. Assume $\deg(\alpha) \leq g-2$ or equivalently, there exist at least two descendents of the point class. We use the degeneration to the normal cone of a smooth elliptic fiber:

$$S \rightsquigarrow S \cup_E (\mathbb{P}^1 \times E).$$

We specialize the point class to the bubble $\mathbb{P}^1 \times E$. Let $C = C' \cup C''$ be the splitting of a domain curve appearing in the degeneration formula in Theorem 5.1. Namely, C' is the component on S and C'' is the component on $\mathbb{P}^1 \times E$. We argue that this splitting has non-trivial contribution only for $g(C') < g$. If $g(C') = g$, this forces C'' to be a disconnected union of two rational curves. Since the degree of the curve class along the divisor is $\langle 2B + hF, F \rangle = 2$, the two descendents of the point class then force the cohomology weighted partition to be $(1, \mathsf{1})^2$ on the bubble or, equivalently, $(1, \omega)^2$ for (S, E). This contribution vanishes because there are no curves which can satisfy this condition (if $(1, \omega)^2$ is represented by a generic point in E^2, see Corollary 5.2).

Case 3. Assume $\deg(\alpha) = g - 1$ or equivalently, there exists only one desecendent of the point class. We may thus assume $\gamma_1 = \mathsf{p}$. If $n = 1, g \geq 2$, we can move $\tau_{g-1}(\mathsf{p})$ to the bubble and the genus on S drops.

When $n \geq 2$, moving the point class to the bubble as in Case 2 may not reduce the genus. In particular, moving $\tau_0(\mathsf{p})$ to the bubble has non-trivial contribution from rational curves on the bubble. On the other hand, if $a \geq 1$, moving $\tau_a(\mathsf{p})$ to the bubble reduces the genus on S because of the dimension constraint.

We use Buryak, Shadrin and Zvonkine's description of the top tautological group $R^{g-1}(M_{g,n})$ [10]. For any $\alpha \in R^{g-1}(\overline{M}_{g,n})$ the restriction of α to $M_{g,n}$ is a linear combination of

$$R^{g-1}(M_{g,n}) = \mathbb{Q}\big\langle \psi_1^{g-1}, \psi_2^{g-1}, \ldots, \psi_n^{g-1} \big\rangle \tag{3.17}$$

and the boundary term is also tautological class in $R^{g-1}(\partial\overline{M}_{g,n})$. By the divisor equation and subsequent use of (3.17), we can reduce to cases for $\leq (g, 2)$. When $g \geq 3$, (3.17) has a different basis

$$R^{g-1}(M_{g,2}) = \mathbb{Q}\big\langle \psi_1^{g-1}, \psi_1\psi_2^{g-2} \big\rangle$$

which is an easy consequence of the generalized top intersection formula. Therefore, we may assume the descendent of the point class is of the form $\tau_a(\mathsf{p})$ with $a \geq 1$. Now, specializing this insertion to the bubble $\mathbb{P}^1 \times E$ reduces the genus and hence the same argument as in Case 2 applies. The genus 2 case is covered by Proposition 6.5.

Relative vs. absolute. We reduced to invariants for (S, E) with genus $g' < g$. As explained in the proof of [29, Lemma 31] (see also [28]), the degeneration formula provides an upper triangular relation between absolute and relative invariants for all pairs $\leq (g', n')$. Thus, our induction applies.

7.2 Proof of Theorem 1.3

We argue by showing that each induction step in the proof of Theorem 1.1 is compatible with the holomorphic anomaly equation. Nontrivial step appears when the degeneration formula is used. From the compatibility result Proposition 5.4, we are reduced to proving the relative holomorphic anomaly equation for lower genus relative generating series $\mathsf{F}^{\text{rel}}_{g',2}$ for (S, E) and relative generating series for $(\mathbb{P}^1 \times E, E)$. The holomorphic anomaly equation for $(\mathbb{P}^1 \times E, E)$ is established in [33]. Because of the relative vs. absolute correspondence [28], we are reduced to proving the holomorphic anomaly equation for $\mathsf{F}_{g',2}$ in genus $0, 1$ and some genus 2 descendents. We proved the multiple cover formula for these cases in Section 6, which implies the holomorphic anomaly equation by Proposition 1.5.

Remark 7.1. Parallel argument shows that we can always reduce the proof for arbitrary descendent insertions to the case when the number of point insertions is less than or equal to $m - 1$.

8 Examples

Example 8.1. We compute $\mathsf{F}_{1,2}\big(\tau_1(F)\big)$ via topological recursion in genus one and illustrate Conjecture 3.1. Let $[\delta_0] \in A^1(\overline{M}_{1,1})$ be the pushforward of the fundamental class under the gluing map

$$\overline{M}_{0,3} \to \overline{M}_{1,1}\,.$$

Since

$$\psi_1 = \frac{1}{24}[\delta_0] \in A^1(\overline{M}_{1,1})\,,$$

we obtain

$$\begin{aligned}\mathsf{F}_{1,1}\big(\tau_1(F)\big) &= \frac{1}{24}\mathsf{F}_{0,1}\big(\tau_0(F)\tau_0(\Delta_S)\big) = \frac{1}{12}\mathsf{F}_{0,1}\big(\tau_0(F)\tau_0(F\times W)\big)\\ &= \frac{1}{12}\mathsf{D}_q\mathsf{F}_{0,1}\,,\end{aligned}$$

where $\Delta_S \subset S\times S$ is the diagonal class. Analogously,

$$\mathsf{F}_{1,2}\left(\tau_1(F)\right) = \frac{1}{24}\,\mathsf{F}_{0,2}\big(\tau_0(F)\tau_0(\Delta_S)\big) = \frac{1}{3}\,\mathsf{D}_q\mathsf{F}_{0,2}\,.$$

Using the multiple cover formula in genus zero

$$\mathsf{F}_{0,2} = \mathsf{T}_2\mathsf{F}_{0,1} + \frac{1023}{8192}\,\mathsf{F}_{0,1}(q^2)\,,$$

we obtain

$$\begin{aligned}\mathsf{F}_{1,2}\big(\tau_1(F)\big) &= \frac{1}{3}\mathsf{D}_q\mathsf{F}_{0,2} = 2\,\mathsf{T}_2\frac{1}{12}\mathsf{D}_q\mathsf{F}_{0,1} + \frac{1023}{1024}\,\mathsf{B}_2\frac{1}{12}\mathsf{D}_q\mathsf{F}_{0,1}\\ &= 2\,\mathsf{T}_2\mathsf{F}_{1,1}\big(\tau_1(F)\big) + (2^0-2^{-10})\,\mathsf{B}_2\mathsf{F}_{1,1}\big(\tau_1(F)\big)\,,\end{aligned}$$

in perfect agreement with Conjecture 3.1 using the formula for $\mathsf{T}_{2,0}$ from Lemma 2.7.

Example 8.2. We compute $\mathsf{F}_{2,2}(\tau_0(\mathsf{p})^2)$ via degeneration formula and verify the multiple cover formula. The first two terms are computed by the classical geometry of K3 surface in [32]. For simplicity we write $\mathsf{F}_{1,2} = \mathsf{F}_{1,2}(\tau_0(\mathsf{p}))$. The relative invariants for (S,E) can be written in terms of absolute invariants:

Lemma 8.3. (i) $\mathsf{F}^{\mathrm{rel}}_{0,2}\big(\emptyset \mid (1,1)^2\big) = 2\mathsf{F}_{0,2}$,

(ii) $\mathsf{F}^{\mathrm{rel}}_{1,2}\big(\emptyset \mid (1,1),(1,\omega)\big) = \mathsf{F}_{1,2} - 2\mathsf{F}_{0,2}\mathsf{D}_q C_2$,

(iii) $\mathsf{F}^{\mathrm{rel}}_{1,2}\big(\emptyset \mid (2,1)\big) = \frac{1}{3}\mathsf{D}_q\mathsf{F}_{0,2} - 4C_2\mathsf{F}_{0,2}$.

Proof. It is a standard computation of the relative vs. absolute correspondence [28]. □

The relative invariants for $(\mathbb{P}^1 \times E, E)$ can be computed by the Gromov–Witten invariants of E.

Lemma 8.4. (i) $\mathsf{G}^{\mathrm{rel}}_{0,1}\big(\tau_0(\mathsf{p}) \mid (1,1)\big) = 1, \quad \mathsf{G}^{\mathrm{rel}}_{0,1}\big(\emptyset \mid (1,\omega)\big) = 1,$

(ii) $\mathsf{G}^{\mathrm{rel}}_{1,1}\big(\tau_0(\mathsf{p}) \mid (1,\omega)\big) = \mathsf{D}_q C_2, \quad \mathsf{G}^{\mathrm{rel}}_{1,1}\big(\tau_0(\mathsf{p})^2 \mid (1,1)\big) = 2\mathsf{D}_q C_2,$

(iii) $\mathsf{G}^{\mathrm{rel}}_{2,1}\big(\tau_0(\mathsf{p})^2 \mid (1,\omega)\big) = (\mathsf{D}_q C_2)^2,$

(iv) $\mathsf{G}^{\mathrm{rel}}_{1,2}\big(\tau_0(\mathsf{p})^2 \mid (2,\omega)\big) = \mathsf{D}_q^2 C_2, \quad \mathsf{G}^{\mathrm{rel}}_{1,2}\big(\tau_0(\mathsf{p})^2 \mid (1,\omega)^2\big) = \mathsf{D}_q^3 C_2.$

Consider the degeneration where two point insertions move to the bubble $\mathbb{P}^1 \times E$. By Theorem 5.1,

$$\begin{aligned}\mathsf{F}_{2,2}\big(\tau_0(\mathsf{p})^2\big) &= \big(\mathsf{F}_{1,2} - 2\mathsf{F}_{0,2}\mathsf{D}_q C_2\big)4\mathsf{D}_q C_2 + \big(\frac{1}{3}\mathsf{D}_q\mathsf{F}_{0,2} - 4C_2\mathsf{F}_{0,2}\big)2\mathsf{D}_q^2 C_2 \\ &\quad + (2\mathsf{F}_{0,2})\frac{1}{2}\big(\mathsf{D}_q^3 C_2 + 4(\mathsf{D}_q C_2)^2\big) \\ &= 36q + 8760q^2 + 754992q^3 + 36694512q^4 + \cdots .\end{aligned}$$

On the other hand, the primitive generating series

$$\mathsf{F}_{2,1}\big(\tau_0(\mathsf{p})^2\big) = \frac{\big(\mathsf{D}_q C_2\big)^2}{\Delta(q)}$$

is computed in [7] and one can apply the multiple cover formula to obtain a candidate for $\mathsf{F}_{2,2}\big(\tau_0(\mathsf{p})^2\big)$. The first few terms of the two generating series match. It is enough to conclude that the two generating series are indeed equal because the space of quasimodular forms with given weight is finite dimensional. However, it seems non-trivial to match the above formula from the degeneration with the formula provided by Conjecture 3.1.

A proof of degeneration formula

For a self-contained exposition, we present a proof of the degeneration formula which is parallel to the proof in [29, 30]. When $m = 1, 2$, a proof using symplectic geometry was presented in [24].

Perfect obstruction theory

For simplicity assume $n = 0$. General cases easily follow from this case. Let $\epsilon\colon \mathcal{S} \to \mathbb{A}^1$ be the total family of the degeneration and

$$\overline{M}_g(\epsilon, \beta) \to \mathbb{A}^1$$

be the moduli space of stable maps to the expanded target $\widetilde{\mathcal{S}}$. For the relative profile μ, the embedding

$$\iota_\mu\colon \overline{M}_g(\mathcal{S}_0, \mu) \hookrightarrow \overline{M}_g(\epsilon, \beta)$$

can be realized as a Cartier pseudo-divisor (L_μ, s_μ).

Let $\mathrm{E}_\epsilon \to \mathbb{L}_{\overline{\mathrm{M}}_g(\epsilon,\beta)}$ be the perfect obstruction theory constructed in [27]. Then the perfect obstruction theories E_0 and E_μ of $\overline{M}_g(\mathcal{S}_0, \beta)$ and $\overline{M}_g(\mathcal{S}_0, \mu)$ sit in exact triangles

$$L_0^\vee \to \iota_0^*\mathrm{E}_\epsilon \to \mathrm{E}_0 \xrightarrow{[1]}$$
$$L_\mu^\vee \to \iota_\mu^*\mathrm{E}_\epsilon \to \mathrm{E}_\mu \xrightarrow{[1]} .$$

On $\overline{M}_g(\mathcal{S}_0, \mu)$, the perfect obstruction theory splits as follows. Let E_1 and E_2 be the perfect obstruction theory of relative stable map spaces $\overline{M}_g(S/E, \beta_1)_\mu$ and $\overline{M}_g(\mathbb{P}^1 \times E/E, \beta_2)_\mu$ respectively. There exists an exact triangle

$$\bigoplus_{i=1}^{l(\mu)} (N^\vee_{\Delta_E/E\times E})_i \to \mathrm{E}_1 \boxplus \mathrm{E}_2 \to \mathrm{E}_\mu \xrightarrow{[1]} \tag{3.18}$$

where $(N^\vee_{\Delta_E/E\times E})_i$ is the pullback of the conormal bundle of the diagonal $\Delta_E \subset E \times E$ along the i-th relative marking.

Reduced class

Let $\rho\colon \widetilde{\mathcal{S}} \to S \times \mathbb{A}^1 \to S$ be the projection. By pulling back the holomorphic symplectic form on S via ρ, one can define a cosection of the obstruction sheaf of E_ϵ

$$Ob_{\overline{M}_g(\epsilon,\beta)} \to \mathcal{O}\,,$$

see [20, Section 5]. Dualizing the cosection gives a morphism

$$\gamma\colon \mathcal{O}[1] \to \mathrm{E}_\epsilon\,.$$

Let $\mathrm{E}_\epsilon^{\mathrm{red}}$ be the cone of γ which gives the reduced class on $\overline{M}_g(\epsilon, \beta)$. Similarly we can construct

$$\gamma_{\mathrm{rel}}\colon \mathcal{O}[1] \to \mathrm{E}_1$$

for the moduli space of relative stable maps $\overline{M}_g(S/E, \beta)$.

Degeneration formula for reduced class

Restricting γ to $\overline{M}_g(\mathcal{S}_0, \beta)$ and $\overline{M}_g(\mathcal{S}_0, \mu)$, we get

$$\begin{aligned} \gamma_0 &\colon \mathcal{O}[1] \to \iota_0^* \mathrm{E}_\epsilon \to \mathrm{E}_0 \\ \gamma_\mu &\colon \mathcal{O}[1] \to \iota_\mu^* \mathrm{E}_\epsilon \to \mathrm{E}_\mu \end{aligned}$$

where the compositions induce reduced classes. The exact triangles

$$\begin{aligned} L_0^\vee &\to \iota_0^* \mathrm{E}_\epsilon^{\mathrm{red}} \to \mathrm{E}_0^{\mathrm{red}} \xrightarrow{[1]}, \\ L_\mu^\vee &\to \iota_\mu \mathrm{E}_\epsilon^{\mathrm{red}} \to \mathrm{E}_\mu^{\mathrm{red}} \xrightarrow{[1]}, \end{aligned}$$

still hold.

Lemma 8.5. We have an exact triangle

$$N^\vee_{\Delta_{E^l}/E^l \times E^l} \to \mathrm{E}_1^{\mathrm{red}} \boxplus \mathrm{E}_2 \to \mathrm{E}_\mu^{\mathrm{red}} \xrightarrow{[1]}$$

on $\overline{M}_g(\mathcal{S}_0, \mu)$ compatible with the structure maps to the cotangent complex.

Proof. Consider the diagram of complexes

$$\begin{array}{ccccc} & & \mathcal{O}[1] \boxplus 0 & = & \mathcal{O}[1] \\ & & \downarrow{\scriptstyle \gamma_{\mathrm{rel}} \boxplus 0} & & \downarrow{\scriptstyle \gamma_\mu} \\ \bigoplus_{i=1}^{l(\mu)} (N^\vee_{\Delta_E/E\times E})_i & \longrightarrow & \mathrm{E}_1 \boxplus \mathrm{E}_2 & \longrightarrow & \mathrm{E}_\mu \\ \| & & \downarrow & & \downarrow \\ \bigoplus_{i=1}^{l(\mu)} (N^\vee_{\Delta_E/E\times E})_i & \longrightarrow & \mathrm{E}_1^{\mathrm{red}} \boxplus \mathrm{E}_2 & \longrightarrow & \mathrm{E}_\mu^{\mathrm{red}} \end{array}$$

where the middle horizontal morphisms are the exact triangle from (3.18). The square on the top commutes because the cosections for $\widetilde{\mathcal{S}}$ and (S, E) are both coming from the holomorphic symplectic form on S. The vertical morphisms are exact triangles and hence induces a map between cones. □

Now Theorem 5.1 is a direct consequence of Lemma 8.5.

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Chapter 4
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Curves on K3 surfaces in divisibility two (with Younghan Bae)
Forum of mathematics, Sigma, 9, e9 (2021)

Motives of moduli spaces on K3 surfaces and of special cubic fourfolds
manuscripta math, 161, 109–124 (2020)

Gromov–Witten classes of K3 surfaces
arXiv:1912.00389 (17 pages)