Motives of hyperkähler varieties and special cubic fourfolds

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Introduction

Given a moduli space M of stable sheaves on a variety S, one expects that certain invariants of M are determined by the geometry of S. This question can be studied on various levels, including derived categories, Hodge structures and algebraic cycles. We will focus on the Chow groups and motives in the case of a K3 surface (or an abelian surface) S.

The analogous question for moduli spaces of stable vector bundles on a curve has been studied by del Baño [22]. He showed that the Chow motive of the moduli space is contained in the full pseudo-abelian tensor subcategory generated by the motive of the curve and the Lefschetz motive. For surfaces, a natural notion of stability for sheaves is provided by Gieseker stability (with respect to some polarization). More generally, we will consider stability for α-twisted sheaves with $\alpha \in \mathrm{Br}(S)$ a Brauer class. The case of a moduli space of Gieseker stable sheaves corresponds to the trivial Brauer class $\alpha = 1$. The first main result of this thesis is:

Theorem 0.1. *Let S be a complex projective K3 surface or an abelian surface and $\alpha \in \mathrm{Br}(S)$. Assume that M is one of the following:*

- *a smooth projective moduli space of Gieseker stable α-twisted sheaves or*
- *a moduli space of σ-stable objects in $\mathrm{D}^{\mathrm{b}}(S,\alpha)$ with primitive Mukai vector and generic stability condition $\sigma \in \mathrm{Stab}^{\dagger}(S,\alpha)$.*

Then the Chow motive $\mathfrak{h}(M)$ of M is a direct summand of a motive $\bigoplus \mathfrak{h}(S^{k_i})(n_i)$ for some $1 \leqslant k_i \leqslant \dim M$, $n_i \in \mathbb{Z}$.

As a direct consequence, finite dimensionality of the motive of S implies the same for M:

Corollary 0.2. *Let S and M be as above. If $\mathfrak{h}(S)$ is finite dimensional, then $\mathfrak{h}(M)$ is finite dimensional as well.* □

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The second half of this thesis has a similar flavour; we investigate the relation between K3 surfaces and cubic fourfolds on the level of algebraic cycles. Recall the divisors $\mathcal{C}_d \subseteq \mathcal{C}$ of smooth complex cubic fourfolds admitting a labelling of discriminant d (see Section 1 for a brief review of the relevant notions). We prove the following result:

Theorem 0.3. *Let $X \in \mathcal{C}_d$ be a special cubic fourfold. Assume that there exist a K3 surface S, a Brauer class $\alpha \in \mathrm{Br}(S)$ and a Hodge isometry $\widetilde{H}(S, \alpha, \mathbb{Z}) \simeq \widetilde{H}(\mathcal{A}_X, \mathbb{Z})$. Then there is a cycle $Z \in \mathrm{CH}^3(S \times X)$ inducing an isomorphism of Chow groups $\mathrm{CH}_0(S)_{\mathrm{hom}} \xrightarrow{\sim} \mathrm{CH}_1(X)_{\mathrm{hom}}$ and transcendental motives $\mathfrak{t}(S)(1) \simeq \mathfrak{t}(X)$.*

A (twisted) K3 surface and a Hodge isometry $\widetilde{H}(S, \alpha, \mathbb{Z}) \simeq \widetilde{H}(\mathcal{A}_X, \mathbb{Z})$ as above exist if and only if d satisfies the numerical condition $(**')$ (see Definition 1.5).

The two results fit into the following picture. For a variety X we denote by $\mathrm{Mot}(X)$ the full pseudo-abelian tensor subcategory of motives generated by $\mathfrak{h}(X)$ and the Lefschetz motive $\mathbb{L}$. Let now X be a cubic fourfold and F its Fano variety of lines, which is a hyperkähler variety of dimension four. It is known that the motive of F is contained in $\mathrm{Mot}(X)$ (we say that $\mathfrak{h}(F)$ is motivated by $\mathfrak{h}(X)$ following Arapura [5]). Indeed, Laterveer proved a formula for Chow motives (which is similar to the result obtained by Galkin–Shinder [26] in the Grothendieck ring of varieties):

Proposition 0.4 (Laterveer [44])**.**

$$\mathfrak{h}(F)(2) \oplus \bigoplus_{i=0}^{4} \mathfrak{h}(X)(i) \simeq \mathfrak{h}(X^{[2]}).$$

Since the Hilbert scheme $X^{[2]}$ can be described as a blow-up of the symmetric product $X^{(2)}$ along the diagonal, its motive is motivated by $\mathfrak{h}(X)$. In Section 2 we will argue that $\mathfrak{h}(X)$ is also motivated by $\mathfrak{h}(F)$ (see also [14, Thm. 4.5]):

Corollary 0.5. *Let X be a cubic fourfold and F its Fano variety of lines. The full pseudo-abelian tensor categories of motives generated by the Lefschetz motive and $\mathfrak{h}(X)$ resp. $\mathfrak{h}(F)$ agree:*

$$\mathrm{Mot}(X) = \mathrm{Mot}(F).$$

In particular, $\mathfrak{h}(X)$ is finite dimensional if and only if $\mathfrak{h}(F)$ is so.

To compare this result with Theorem 0.1, assume that X is a special cubic fourfold satisfying condition $(**')$. This is equivalent to the Fano variety F being birational to a moduli space M of stable twisted sheaves on a K3 surface S (cf. [33, Prop. 4.1]). In this case, all of the following categories of motives agree:

$$\mathrm{Mot}(S) = \mathrm{Mot}(M) = \mathrm{Mot}(F) = \mathrm{Mot}(X).$$

Indeed, we know that birational hyperkähler varieties have isomorphic Chow motives (see e.g. Theorem 2.14). This induces the middle equality. It follows from Theorem 0.3 that $\mathrm{Mot}(S) = \mathrm{Mot}(X)$. For an arbitrary complex projective K3 surface and a moduli space M as in Theorem 0.1 we have at least an inclusion $\mathrm{Mot}(M) \subseteq \mathrm{Mot}(S)$ which we expect to be an equality as well (see Remark 2.28 for some comments).

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Notations and Conventions. We will work over the complex numbers unless otherwise stated. The bounded derived category of coherent sheaves on a smooth projective variety X is denoted by $\mathrm{D}^{\mathrm{b}}(X)$. Morphisms between triangulated categories are assumed to be exact and $\mathbb{C}$-linear. Throughout this thesis, all motives are meant to be Chow motives in the sense of Subsection 2.1.

1. Cubic fourfolds and their Fano varieties

We review the main facts about cubic fourfolds and the connection to their Fano varieties of lines. The recent survey article [29] provides a more detailed account. See also [10]. It is a classical (and certainly very difficult) problem to determine which cubic fourfolds are rational and we will not touch upon this question at all. Instead, we aim to study the connection between special cubic fourfolds and K3 surfaces on the level of cycles.

1.1. **General facts on cubic fourfolds.** A cubic fourfold is a smooth complex cubic hypersurface $X \subseteq \mathbb{P}^5$. The moduli stack of cubic fourfolds is a Deligne–Mumford stack with coarse moduli space $\mathcal{C}$. It is a quasi-projective variety of dimension 20 and will be important in the sequel. One can describe $\mathcal{C}$ as a GIT quotient (see [51, Ch. 4.2]):

$$\mathcal{C} = \mathcal{U}/\mathrm{PGL}_6,$$

where $\mathcal{U}$ is the open subset of the complete linear system $|\mathcal{O}_{\mathbb{P}^5}(3)|$ parametrizing smooth cubic fourfolds.

The cohomology of a cubic fourfold X is well understood (see e.g. [10]); it is torsion-free and the primitive cohomology sits in degree four. Its particular shape suggests that a part of the Hodge structure comes from a K3 surface (see [57] or [29, Section 3.1] for more details behind this philosophy). The integral Hodge conjecture holds for cubic fourfolds [67, Thm. 18] and the cycle class map is injective [18, Prop. 5.1] in codimension two. Therefore

$$\mathrm{CH}^2(X) \xrightarrow{\sim} H^{2,2}(X, \mathbb{Z})$$

and if X is very general, any surface is homologous to a multiple of h^2, with $h = c_1(\mathcal{O}_X(1))$ the class of a hyperplane section. Thus, $\mathrm{rk}(H^{2,2}(X,\mathbb{Z})) = 1$ in this case.

Definition 1.1. A cubic fourfold X is called special if $\mathrm{rk}(H^{2,2}(X,\mathbb{Z})) \geqslant 2$. A labelling is a primitive sublattice $K \subseteq H^{2,2}(X,\mathbb{Z})$ of rank two containing h^2. Its discriminant is the determinant of the intersection form on K. One defines $\mathcal{C}_d$ as the set of cubic fourfolds admitting a labelling of discriminant d.

The sets $\mathcal{C}_d$ have been introduced by Hassett who proved the following result (cf. [28, Thm. 1.0.1]):

Theorem 1.2 (Hassett). *For each d the set $\mathcal{C}_d \subseteq \mathcal{C}$ is an irreducible divisor. It is non-empty if and only if $d > 6$ and $d \equiv 0, 2 \mod 6$.*

The existence of a labelling of discriminant d (for certain values of d) has very concrete consequences for the Hodge structure resp. derived category of X and the geometry of its Fano variety as will be explained later on. For the moment, we only recall a theorem of Hassett (cf. [28, Thm. 1.0.2]) which gives the connection between cubic fourfolds and K3 surfaces on the level of Hodge structures:

Theorem 1.3 (Hassett). *Let X be a cubic fourfold. There exist a labelling $K \subseteq H^{2,2}(X,\mathbb{Z})$, a polarized K3 surface S and a Hodge isometry $H^2(S,\mathbb{Z})_{\mathrm{pr}}(-1) \simeq K^{\perp}$ if and only if $X \in \mathcal{C}_d$ with d not divisible by 4, 9 or any odd prime $p \equiv 2 \mod 3$.*

1.2. **Kuznetsov component of the derived category.** Fix a cubic fourfold X. It was observed by Kuznetsov [40] that the derived category of X admits a distinguished semi-orthogonal decomposition:

$$\mathrm{D^b}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle.$$

Kuznetsov investigated the category $\mathcal{A}_X$ in detail and found that it behaves similarly to the derived category of a K3 surface. Indeed, its Serre functor is a double shift [2] and its Hochschild homology is isomorphic to the one of a K3 surface. He conjectured that X is rational if and only if $\mathrm{D^b}(S) \simeq \mathcal{A}_X$ for some projective K3 surface S.

Addington and Thomas introduced a lattice $\widetilde{H}(\mathcal{A}_X,\mathbb{Z})$ of rank 24 for $\mathcal{A}_X$, which comes with a natural weight two Hodge structure (see [1, Section 2]). Roughly, it is the orthogonal complement of the classes of $\mathcal{O}_X$, $\mathcal{O}_X(1)$, $\mathcal{O}_X(2)$ in $\mathrm{K_{top}}(X)$ with respect to the Euler pairing. The Hodge structure is pulled back from $H^*(X,\mathbb{Z})$ via the Mukai vector. Abstractly, the lattice $\widetilde{H}(\mathcal{A}_X,\mathbb{Z})$ is isomorphic to the even, unimodular lattice $E_8^{\oplus 2}(-1) \oplus \mathrm{U}^{\oplus 4}$ with signature $(4, 20)$, where U is the hyperbolic plane. It is an analogue of the Mukai lattice of a K3 surface S and, in fact, it agrees with it in the case $\mathrm{D^b}(S) \simeq \mathcal{A}_X$.

Remark 1.4. Be aware that (with our convention) the Mukai pairing on a K3 surface satisfies $(v(\mathcal{E}), v(\mathcal{F})) = -\chi(\mathcal{E},\mathcal{F})$, and similarly the bilinear form on $\widetilde{H}(\mathcal{A}_X,\mathbb{Z})$ is given by $-\chi$.

1.3. **The Fano variety of lines.** Given a cubic fourfold $X \subseteq \mathbb{P}^5$, its Fano variety of lines $F(X)$ is a smooth projective variety of dimension four. It was shown by Beauville and Donagi [10] that $F(X)$ is a hyperkähler variety, i.e. it is simply connected and admits a (up to scalars) unique, everywhere non-degenerate holomorphic two-form (see also [32, Ch. 4] for examples of hyperkähler varieties). Moreover, the universal line $L \subseteq F(X) \times X$ induces a Hodge isometry, known as the *Abel–Jabobi map*, between the primitive cohomologies

$$H^4(X,\mathbb{Z})_{\mathrm{pr}} \xrightarrow{\sim} H^2(F(X),\mathbb{Z})_{\mathrm{pr}}(-1).$$

The polarization on $F(X)$ is given via the Plücker embedding and $H^2(F(X),\mathbb{Z})$ is equipped with the *Beauville–Bogomolov form* (see e.g. [32, Ch. 5]).

1.4. **Associated K3 surfaces.** Hassett described how in certain cases a K3 surface is associated to a cubic fourfold X on the level of Hodge structures. In fact, there are other ways a K3 surface can be associated to X and we try to give an overview with the table below. To characterize special cubic fourfolds contained in a divisor $\mathcal{C}_d$ we will use the following numerical conditions on d:

Definition 1.5.

$$\exists a, n \in \mathbb{Z} : a^2 d = 2n^2 + 2n + 2, \qquad (***)$$

$$\exists n \in \mathbb{Z} : d \mid 2n^2 + 2n + 2, \qquad (**)$$

$$\exists k, d_0 \in \mathbb{Z} : d_0 \text{ satisfies } (**) \text{ and } d = k^2 d_0. \qquad (**')$$

Remark 1.6. Alternatively, the conditions may be called (K3$^{[2]}$), (K3) resp. (K3′) referring to their geometric meaning (see table below). It can be shown that $(**)$ is equivalent to Hassett's condition (Theorem 1.3).

There are (strict) inclusions of dense subsets inside the moduli space $\mathcal{C}$ of cubic fourfolds:

$$\bigcup_{(***)} \mathcal{C}_d \quad \subseteq \quad \bigcup_{(**)} \mathcal{C}_d \quad \subseteq \quad \bigcup_{(**')} \mathcal{C}_d.$$

As a culmination of results (see [1, 3, 7, 33]) these sets are characterized in several equivalent ways. In each case, the condition means that X satisfies $(**)$ resp. $(**')$ if and only if there exists some K3 surface with the given property. We use the following notation:

- X is a special cubic fourfold with a labelling $K \subseteq H^{2,2}(X,\mathbb{Z})$,
- S is a projective K3 surface,
- $T(S)$ is the orthogonal complement of $H^{1,1}(S,\mathbb{Z})$ in $H^2(S,\mathbb{Z})$,
- $T(X)$ is the orthogonal complement of $H^{2,2}(X,\mathbb{Z})$ in $H^4(X,\mathbb{Z})$,
- $\alpha \in \mathrm{Br}(S)$ is a Brauer class,
- $M_H(v,\alpha)$ is a moduli space of Gieseker stable α-twisted sheaves with Mukai vector v on S. Here, we consider Gieseker stability with respect to an ample line bundle H which

is generic for v. See Subsection 2.4 for a brief review. If the Brauer class vanishes, we omit it from the notation.

- $M_\sigma(v,\alpha)$ is a moduli space of σ-stable objects in $\mathrm{D^b}(S,\alpha)$ with Mukai vector v, where $\sigma \in \mathrm{Stab}^\dagger(S,\alpha)$ is a generic stability condition. If the Brauer class vanishes, we omit it from the notation.

	$\bigcup_{(**)} \mathcal{C}_d$	$\bigcup_{(**')} \mathcal{C}_d$
Hodge theory	$H^2(S,\mathbb{Z})_{\mathrm{pr}}(-1) \simeq K^\perp$	
$\Longleftrightarrow$	$T(S)(-1) \simeq T(X)$	
$\Longleftrightarrow$	$\widetilde{H}(S,\mathbb{Z}) \simeq \widetilde{H}(\mathcal{A}_X,\mathbb{Z})$	$\widetilde{H}(S,\alpha,\mathbb{Z}) \simeq \widetilde{H}(\mathcal{A}_X,\mathbb{Z})$
Derived category[1]	$\mathrm{D^b}(S) \simeq \mathcal{A}_X$	$\mathrm{D^b}(S,\alpha) \simeq \mathcal{A}_X$
Hyperkähler	$M_H(v) \sim F(X)$ birational	$M_H(v,\alpha) \sim F(X)$ birational
$\Longleftrightarrow$	$M_\sigma(v) \simeq F(X)$	$M_\sigma(v,\alpha) \simeq F(X)$

Remark 1.7. For the condition $(***)$ there is only one notable equivalent characterization, namely that the Fano variety $F(X)$ is birational to the Hilbert scheme of a K3 surface S:

$$X \in \mathcal{C}_d \text{ for some } d \text{ satisfying } (***) \Longleftrightarrow F(X) \sim S^{[2]}.$$

2. Motives of hyperkähler varieties

This section is divided into two parts. The first half serves as a recollection of the basic facts about Chow groups and motives. Since the literature is vast (see e.g. [52]), we will review only parts of the general theory and focus on the motives of K3 surfaces and cubic fourfolds. We also include some remarks on the notion of finite dimensionality in the sense of Kimura and O'Sullivan. The second half deals with motives of hyperkähler varieties and contains the proof of Theorem 0.1.

2.1. **Motives.** Let X be a smooth projective variety. The Chow groups $\mathrm{CH}^i(X)$ (resp. $\mathrm{CH}_i(X)$) are constructed by using codimension (resp. dimension) i algebraic cycles modulo rational equivalence. When working with rational coefficients, we use a subscript '$\mathbb{Q}$'. The homologically trivial cycles (with respect to some fixed Weil cohomology theory) are denoted with a subscript 'hom'.

Definition 2.1. The objects of the category $\mathrm{Mot}_{\mathbb{C}}$ of (Chow) motives are triples (X,p,m), with X a smooth projective variety over $\mathbb{C}$, $p \in \mathrm{CH}^{\dim(X)}(X \times X)_{\mathbb{Q}}$ a projector (with respect to

[1]At the moment, an equivalence $\mathrm{D^b}(S) \simeq \mathcal{A}_X$ (resp. $\mathrm{D^b}(S,\alpha) \simeq \mathcal{A}_X$) is established only for generic $X \in \mathcal{C}_d$. This gap is expected to be filled soon (see the upcoming work of Bayer, Lahoz, Macrì, Nuer, Perry, Stellari [6]).

convolution) and m an integer. Morphisms are defined by

$$\mathrm{Hom}((X,p,m),(Y,q,n)) = q \circ \mathrm{CH}^{\dim(X)+n-m}(X \times Y)_{\mathbb{Q}} \circ p.$$

The motive $\mathfrak{h}(X) = (X, [\Delta_X], 0)$ is called the motive of X. Define the Lefschetz motive $\mathbb{L} = (\mathbb{P}^1, [\mathbb{P}^1 \times \mathrm{pt}], 0) \simeq (\mathrm{Spec}(\mathbb{C}), [\Delta], -1)$ and the motive of a point $\mathbb{1} = \mathfrak{h}(\mathrm{Spec}(\mathbb{C}))$.

Remark 2.2. The category $\mathrm{Mot}_{\mathbb{C}}$ is pseudo-abelian by construction. Defining the product $\mathfrak{h}(X) \otimes \mathfrak{h}(Y) = \mathfrak{h}(X \times Y)$ for smooth projective varieties X, Y gives $\mathrm{Mot}_{\mathbb{C}}$ the structure of a rigid tensor category. See [52] for details. One writes $M(k)$ for $M \otimes \mathbb{L}^k$.

To each motive $M = (X, p, m)$ one can associate its Chow and cohomology groups

$$\mathrm{CH}^i(M) = \mathrm{Hom}(\mathbb{L}^i, M) = p_* \mathrm{CH}^{i+m}(X)_{\mathbb{Q}} \text{ and } H^i(M) = p_* H^{i+m}(X).$$

Here, H is any Weil cohomology theory; we will use singular cohomology with rational coefficients. It is a natural question whether the decomposition $H^*(M) = \bigoplus H^i(M)$ can be lifted to the level of motives. More precisely, the Chow–Künneth conjecture asks for a decomposition of the diagonal by algebraic projectors π^i inducing the projection on the i-th summand on cohomology. The corresponding isomorphism of motives $\mathfrak{h}(X) \simeq \bigoplus \mathfrak{h}^i(X)$, with $\mathfrak{h}^i(X) = (X, \pi^i, 0)$, is called a Chow–Künneth decomposition. Such projectors exist for surfaces and complete intersections (cf. [52, Ch. 6, Appendix C]). The following refinements will be important later on:

Theorem 2.3. *Let S be a projective K3 surface and X a cubic fourfold. Then the Chow–Künneth conjcture holds for S and X. More precisely, there are equations in* $\mathrm{CH}^2(S \times S)_{\mathbb{Q}}$ *and* $\mathrm{CH}^2(X \times X)_{\mathbb{Q}}$*:*

$$[\Delta_S] = \pi_S^0 + \pi_S^{2,\mathrm{alg}} + \pi_S^{2,\mathrm{tr}} + \pi_S^4,$$
$$[\Delta_X] = \pi_X^0 + \pi_X^2 + \pi_X^{4,\mathrm{alg}} + \pi_X^{4,\mathrm{tr}} + \pi_X^6 + \pi_X^8,$$

inducing (refined) Chow–Künneth decompositions

$$\mathfrak{h}(S) \simeq \mathbb{1} \oplus \mathfrak{h}_S^{2,\mathrm{alg}} \oplus \mathfrak{t}(S) \oplus \mathbb{L}^2,$$
$$\mathfrak{h}(X) \simeq \mathbb{1} \oplus \mathbb{L} \oplus \mathfrak{h}_X^{4,\mathrm{alg}} \oplus \mathfrak{t}(X) \oplus \mathbb{L}^3 \oplus \mathbb{L}^4.$$

The motives $\mathfrak{t}(S) = (S, \pi_S^{2,\mathrm{tr}}, 0)$ and $\mathfrak{t}(X) = (X, \pi_X^{4,\mathrm{tr}}, 0)$ are called the transcendental motives of S and X. Moreover, $\mathfrak{h}_S^{2,\mathrm{alg}} \simeq \mathbb{L}^{\oplus \rho}$, $\mathfrak{h}_X^{4,\mathrm{alg}} \simeq (\mathbb{L}^2)^{\oplus \rho_2}$ with ρ the Picard rank of S and $\rho_2 = \dim H^{2,2}(X, \mathbb{Q})$. The Chow groups are given by:

	$\mathfrak{h}_S^{2,\mathrm{alg}}$	$\mathfrak{t}(S)$	$\mathfrak{h}_X^{4,\mathrm{alg}}$	$\mathfrak{t}(X)$
CH^0				
CH^1	$\mathrm{NS}(S)_{\mathbb{Q}}$			
CH^2		$\mathrm{CH}_0(S)_{\mathrm{hom},\mathbb{Q}}$	$\mathrm{CH}^2(X)_{\mathbb{Q}}$	
CH^3				$\mathrm{CH}_1(X)_{\mathrm{hom},\mathbb{Q}}$.
CH^4				

Proof. The refined Chow–Künneth decomposition for surfaces is defined in [52, Ch. 6] and in the case of a K3 surface the Albanese and Picard motives vanish. Let $\ell_1, \dots, \ell_\rho$ be an orthogonal basis of $\mathrm{CH}^1(S)_{\mathbb{Q}} \simeq \mathrm{NS}(S)_{\mathbb{Q}}$. Define the following projectors:

$$\pi_S^0 = [\mathrm{pt} \times S], \quad \pi_S^{2,\mathrm{alg}} = \sum_{i=1}^{\rho} \frac{\ell_i \times \ell_i}{(\ell_i)^2}, \quad \pi_S^4 = [S \times \mathrm{pt}], \quad \pi_S^{2,\mathrm{tr}} = [\Delta_S] - \pi_S^0 - \pi_S^{2,\mathrm{alg}} - \pi_S^4.$$

For cubic fourfolds one proceeds similarly (cf. [56, Sec. 4]). Let $f_1, \dots, f_{\rho_2}$ be an orthogonal basis of $\mathrm{CH}^2(X)_{\mathbb{Q}} \simeq H^{2,2}(X, \mathbb{Q})$. Let $h = c_1(\mathcal{O}_X(1))$ be the class of a hyperplane section. Define the following projectors:

$$\pi_X^0 = [\mathrm{pt} \times X], \quad \pi_X^2 = \frac{1}{3}[h^3 \times h], \quad \pi_X^{4,\mathrm{alg}} = \sum_{i=1}^{\rho_2} \frac{f_i \times f_i}{(f_i)^2}, \quad \pi_X^6 = \frac{1}{3}[h \times h^3], \quad \pi_X^8 = [X \times \mathrm{pt}],$$

$$\pi_X^{4,\mathrm{tr}} = [\Delta_X] - \pi_X^0 - \pi_X^2 - \pi_X^{4,\mathrm{alg}} - \pi_X^6 - \pi_X^8.$$

The remaining assertions are easily verified (see loc. cit.). □

Remark 2.4. The cohomology groups of the transcendental motives are given by:

$$H^*(\mathfrak{t}(S)) = H^2(\mathfrak{t}(S)) = T(S)_{\mathbb{Q}} \text{ and } H^*(\mathfrak{t}(X)) = H^4(\mathfrak{t}(X)) = T(X)_{\mathbb{Q}},$$

where $T(S)$ and $T(X)$ are the transcendental lattices.

Remark 2.5. One can also consider the following (coarser) decomposition of the motive of a cubic fourfold X, which will be used in the proof of Theorem 0.3. As above, let $h \in \mathrm{CH}^1(X)$ be the class of a hyperplane section. Define the primitive projector $\pi_X^{\mathrm{pr}} = [\Delta_X] - \pi_X^0 - \pi_X^2 - \frac{1}{3}[h^2 \times h^2] - \pi_X^6 - \pi_X^8$ and the *primitive motive* $\mathfrak{h}^{\mathrm{pr}}(X) = (X, \pi_X^{\mathrm{pr}}, 0)$. There is a decomposition:

$$\mathfrak{h}(X) \simeq \mathbb{1} \oplus \mathbb{L} \oplus \mathbb{L}^2 \oplus \mathfrak{h}^{\mathrm{pr}}(X) \oplus \mathbb{L}^3 \oplus \mathbb{L}^4.$$

Finally, let us recall the notion of finite dimensionality in the sense of Kimura and O'Sullivan for motives $M = (X, p, m) \in \mathrm{Mot}_{\mathbb{C}}$. Note that the symmetric group $\mathfrak{S}_k$ acts on X^k by permutation and there are associated projectors $\mathrm{alt}_k = \frac{1}{k!}\sum_{\pi \in \mathfrak{S}_k} \mathrm{sgn}(\pi)[\Gamma_\pi]$ resp. $\mathrm{sym}_k = \frac{1}{k!}\sum_{\pi \in \mathfrak{S}_k}[\Gamma_\pi]$. One defines the following two series of motives attached to M:

Definition 2.6. The motives

$$\mathrm{Alt}^k(M) = (X^k, \mathrm{alt}_k \circ p^k, km) \text{ and } \mathrm{Sym}^k(M) = (X^k, \mathrm{sym}_k \circ p^k, km)$$

are called the k-th *alternating* resp. *symmetric product* of M.

The motive $\mathrm{Alt}^k(M)$ is the motivic analogue of the usual alternating product of $H^*(M)$ in the case when M has only even cohomology. Analogously, $\mathrm{Sym}^k(M)$ is the analogue of the alternating product if M has only odd cohomology and we consider $H^*(M)$ just as a vector space (without grading). The missing signs in sym_k take care of the fact that $H^*\colon \mathrm{Mot}_{\mathbb{C}} \longrightarrow \mathrm{grVect}_{\mathbb{Q}}$ is not a tensor functor (cf. [22, Sec. 3.3.2.] for a discussion of this issue). This motivates the following definition of finite dimensionality:

Definition 2.7. Let $M = (X, p, m) \in \mathrm{Mot}_{\mathbb{C}}$ be a Chow motive. We say that M is

(1) *even finite dimensional*, if $\mathrm{Alt}^k(M) = 0$ for $k \gg 0$,
(2) *odd finite dimensional*, if $\mathrm{Sym}^k(M) = 0$ for $k \gg 0$,
(3) *finite dimensional*, if there exist even resp. odd finite dimensional motives M^+ and M^- such that $M \simeq M^+ \oplus M^-$.

Conjecture 2.8 (Kimura–O'Sullivan)**.** All motives $M \in \mathrm{Mot}_{\mathbb{C}}$ are finite dimensional.

The finite dimensionality conjecture has many interesting consequences and it fits well into the web of standard conjectures. Unfortunately, up to now the class of motives which are proven to be finite dimensional is very restricted. It includes for example curves and abelian varieties (cf. [52, Ch. 4.6, Thm. 2.7.2]). The full pseudo-abelian tensor subcategory of $\mathrm{Mot}_{\mathbb{C}}$ generated by motives of abelian varieties consists of finite dimensional motives and we will call them of *abelian type* (following Vial [66]).

We list a few standard results that will be useful. A morphism of motives $f\colon M \longrightarrow N$ is called surjective, if the induced map $\mathrm{CH}^*(M \otimes \mathfrak{h}(Z)) \longrightarrow \mathrm{CH}^*(N \otimes \mathfrak{h}(Z))$ is surjective for all smooth projective varieties Z(cf. [52, Sec. 5.4]). Equivalently, f admits a right inverse and N becomes a direct summand of M (cf. [52, Ex. 2.3.(vii), Lem. 5.4.3]). Note that it suffices to check surjectivity of $\mathrm{CH}^i(M_K) \longrightarrow \mathrm{CH}^i(N_K)$ for all function fields:

Lemma 2.9. *Let $M = (X, p, m)$, $N = (Y, q, n) \in \mathrm{Mot}_{\mathbb{C}}$ and $f \in \mathrm{Hom}(M, N)$ a morphism of motives. Assume that $(f_K)_*\colon \mathrm{CH}^i(M_K) \longrightarrow \mathrm{CH}^i(N_K)$ is surjective for all finitely generated field extensions $\mathbb{C} \subseteq K$. Then f is surjective.*

Proof. Let Z be any variety over $\mathbb{C}$. The proof proceeds by induction on the dimension of Z, the case of dimension zero being trivial. Let K be the function field of Z and $\gamma \in \mathrm{CH}^i(N \otimes \mathfrak{h}(Z))$. We write $\gamma|_{N_K}$ for the pullback of γ to N_K. By assumption, there exists $\delta \in \mathrm{CH}^i(M_K)$ such that $(f_K)_*\delta = \gamma|_{N_K}$. Denote by $\bar{\delta}$ the closure of δ in $X \times Z$. Then $\gamma - (f_Z)_*\bar{\delta}$ is supported on $Y \times Z'$ for some closed proper subvariety $Z' \subseteq Z$ and we conclude by induction. □

Remark 2.10. In fact, it suffices to check surjectivity over $\mathbb{C}$ (see [66, Lem. 3.2]).

Proposition 2.11 (Kimura [37]). *Let $M \longrightarrow N$ be a surjective morphism of motives. If M is finite dimensional, then N is finite dimensional. If $M \simeq M_1 \oplus M_2$, then M_1 and M_2 are finite dimensional if and only if M is finite dimensional. Moreover, if $X \longrightarrow Y$ is a dominant morphism of smooth projective varieties and $\mathfrak{h}(X)$ is finite dimensional, then so is $\mathfrak{h}(Y)$.* □

See [52, Thm. 5.1.4] resp. [52, Ex. 2.8.1(2)] for proofs of the following results.

Lemma 2.12. *Let M and N be finite dimensional motives. Then $M \otimes N$ is finite dimensional as well.* □

Lemma 2.13. *Let X be a smooth projective variety and $Z \subseteq X$ a smooth subvariety of codimension c. Then*

$$\mathfrak{h}(\mathrm{Bl}_Z(X)) \simeq \mathfrak{h}(X) \oplus \bigoplus_{i=1}^{c-1} \mathfrak{h}(Z)(i).$$

Thus, $\mathfrak{h}(\mathrm{Bl}_Z(X))$ is finite dimensional if and only if $\mathfrak{h}(X)$ and $\mathfrak{h}(Z)$ are finite dimensional. □

2.2. **Hyperkähler varieties.** A hyperkähler variety, or irreducible holomorphic symplectic variety, is a simply connected, smooth projective variety X over $\mathbb{C}$ such that $H^0(X, \Omega_X^2)$ is spanned by a everywhere non-degenerate holomorphic two-form. This implies that $\dim(X)$ is even. Hyperkähler varieties of dimension two are precisely K3 surfaces. The natural generalization in higher dimensions are Hilbert schemes $S^{[n]}$ of length n subschemes on a K3 surface S. It was shown by Beauville [9] that they are indeed hyperkähler varieties of dimension $2n$. In fact, all examples of hyperkähler varieties we will consider in this thesis are deformation equivalent to $S^{[n]}$, i.e. they are of K3$^{[n]}$-type. A major class of examples is provided by moduli spaces of stable sheaves on K3 surfaces (see Subsection 2.4 for more details).

Studying the motives of hyperkähler varieties has a particularly nice feature: For two birational hyperkähler varieties X and X' one can always find families $\mathcal{X}$ and $\mathcal{X}'$ over a smooth quasi-projective curve C, which are isomorphic away from a point $0 \in C$ with central fibres $X = \mathcal{X}_0$ resp. $X' = \mathcal{X}'_0$ (cf. [32, Thm. 10.12] and [58, Prop. 2.1]). This can be used to show that their Chow rings $\mathrm{CH}^*(X)$ and $\mathrm{CH}^*(X')$ are isomorphic. The same proof also shows that their Chow motives are isomorphic (see also [63, Sec. 1.6]):

Theorem 2.14. *Let X and X' be birational hyperkähler varieties. There is an isomorphism of Chow motives*

$$\mathfrak{h}(X) \simeq \mathfrak{h}(X').$$

Proof. The following elegant argument is due to Rieß (cf. [58]). Let $\mathcal{X}$, $\mathcal{X}'$ be as above, η the generic point of C and K its function field. Denote by $[\Gamma_\eta] \in \mathrm{CH}(\mathcal{X}_\eta \times_K \mathcal{X}'_\eta)$ the class of the graph of an isomorphism $\mathcal{X}_\eta \simeq \mathcal{X}'_\eta$. The key idea is to use specialization for Chow groups to obtain cycles on $X \times X'$ inducing an isomorphism of motives. More precisely, let $\sigma\colon \mathrm{CH}(\mathcal{X}_\eta \times_K \mathcal{X}'_\eta) \longrightarrow \mathrm{CH}(\mathcal{X}_0 \times \mathcal{X}'_0)$ be the specialization map, which is constructed by taking the closure of a cycle and then restricting to the special fibre. It is compatible with intersection product, pullback and proper pushforward (cf. [25, Ch. 10.1]). Thus,

$$\sigma([{}^{\mathrm{t}}\Gamma_\eta]) \circ \sigma([\Gamma_\eta]) = \sigma([{}^{\mathrm{t}}\Gamma_\eta] \circ [\Gamma_\eta]) = \sigma([\Delta_{\mathcal{X}_\eta}]) = [\Delta_X]$$

and similarly $\sigma([\Gamma_\eta]) \circ \sigma([{}^{\mathrm{t}}\Gamma_\eta]) = [\Delta_{X'}]$. Here, $\circ$ denotes the convolution of cycles. □

Remark 2.15. Note that the multiplicativity of the isomorphism can be expressed as an identity of cycles using the diagonal on $\mathcal{X}_\eta \times_K \mathcal{X}_\eta \times_K \mathcal{X}_\eta$. Specialization yields that $\mathrm{CH}^*(X) \simeq \mathrm{CH}^*(X')$ is actually an isomorphism of graded rings.

2.3. **K3 surfaces.** This subsection contains examples of K3 surfaces with finite dimensional motive. In all cases, a geometric construction allows one to reduce to the case of an abelian variety, where finite dimensionality is known. In fact, the Hodge conjecture would imply that the motive of a K3 surface is always of abelian type (at least in the category of motives up to homological equivalence; the finite dimensionality conjecture would then give the result for $\mathrm{Mot}_{\mathbb{C}}$). This can be deduced from the corresponding result by André ([4, Thm. 7.1]) for so called *motivated motives*, which relies on the Kuga–Satake construction.

Recall that a Nikulin involution ι of a K3 surface S is an automorphism of order two, acting trivially on $H^{2,0}(S)$. The minimal resolution S' of the quotient S/ι is again a K3 surface. We say that S admits a Shioda–Inose structure, if there exists a Nikulin involution, such that S' is a Kummer surface with $T(S)(2) \simeq T(S')$. This implies $\rho(S) \geqslant 17$ since the Picard rank of a Kummer surface is at least 17. The classification of K3 surfaces admitting such a structure is due to Morrison [49].

Proposition 2.16. *Let S be a projective K3 surface. Then S admits a Shioda–Inose structure if and only if one of the following conditions is satisfied:*

(1) $\rho(S) = 19$ *or* 20*,*
(2) $\rho(S) = 18$ *and* $\mathrm{T}(S) \simeq \mathrm{U} \oplus \mathrm{T}'$*, for some lattice* T'*,*
(3) $\rho(S) = 17$ *and* $\mathrm{T}(S) \simeq \mathrm{U}^{\oplus 2} \oplus \mathrm{T}'$*, for some lattice* T'*.*

In each case, $\mathfrak{h}(S)$ *is of abelian type; in particular it is finite dimensional.*

Proof. For the classification result see [49, Cor. 6.4]. A Kummer surface is dominated by the blow-up of an abelian surface, thus its motive is of abelian type by Proposition 2.11. Consequently, if S admits a Shioda–Inose structure, the quotient S/ι has finite dimensional motive. It is known that symplectic involutions of K3 surfaces act trivially on CH^2 (see [69]). Hence, $\mathfrak{t}(S) \simeq \mathfrak{t}(S/\iota)$ and $\mathfrak{h}(S)$ is finite dimensional. □

The same technique applies to certain K3 surfaces which are given as intersections of three quadrics in $\mathbb{P}^5$. They form a four-dimensional family and come with a symplectic action of the group $(\mathbb{Z}/2\mathbb{Z})^4$. The quotient (of a generic member of the family) is a double cover of $\mathbb{P}^2$

branched along six lines. A concrete geometric description due to Paranjape (cf. [54]) implies that its motive is finite dimensional.

Proposition 2.17 (Laterveer [41])**.** *Let $S \subseteq \mathbb{P}^5$ be a K3 surface given by three quadratic equations*

$$a_0x_0^2 + \ldots + a_5x_5^2 = 0,$$
$$b_0x_0^2 + \ldots + b_5x_5^2 = 0,$$
$$c_0x_0^2 + \ldots + c_5x_5^2 = 0,$$

with $a_i, b_i, c_i \in \mathbb{C}$. Then $\mathfrak{h}(S)$ is finite dimensional. □

Remark 2.18. Another family of K3 surfaces with finite dimensional motives may be obtained in view of Theorem 0.3: A recent result of Laterveer (see [42, Thm. 3.1]) building on a construction due to van Geemen–Izadi established finite dimensionality for a 10-dimensional family of cubic fourfolds. More precisely, he proved that $\mathfrak{h}(X)$ is finite dimensional if $X \subseteq \mathbb{P}^5$ is given by an equation $f(x_0, \ldots, x_4) + x_5^3$ such that f defines a smooth cubic threefold (this family has also been studied in [12, Ex. 6.4] and [13]). If S is an associated (twisted) K3 surface of a cubic fourfold X as above, it follows by Corollary 0.5 that $\mathfrak{h}(S)$ is finite dimensional. It remains to describe which K3 surfaces are associated.

Remark 2.19. There are also some sporadic examples of K3 surfaces (of even Picard rank) with finite dimensional motives (cf. [45, Thm. 1, Thm. 2]). They come with a non-symplectic group acting trivially on algebraic cycles and are always dominated by some Fermat surface F_n ([45, Thm. 5]). The geometry of Fermat varieties was studied by Shioda and Katsura [64]. It follows from their results that $\mathfrak{h}(\mathrm{F}_n)$ is of abelian type and one concludes using Proposition 2.11.

2.4. **Moduli spaces of stable sheaves on K3 surfaces.** Let S be a projective K3 surface, $v \in H^*_{\mathrm{alg}}(S, \mathbb{Z})$ primitive and H an ample line bundle on S which is generic with respect to v. The moduli stack of H-Gieseker stable sheaves with Mukai vector v is known to admit a coarse moduli space, which is a hyperkähler variety of dimension $(v)^2 + 2$. See [35] for the general theory of moduli spaces of sheaves. We will also consider moduli spaces $M_\sigma(v)$, where $\sigma \in \mathrm{Stab}^\dagger(S)$ is a generic stability condition (in the sense of Bridgeland [15]) with respect to v, i.e. not contained in a wall for the wall and chamber structure associated to v (cf. [16, Sec. 9]). Toda showed that the moduli stack of σ-semistable objects in $\mathrm{D^b}(S)$ with Mukai vector v (and fixed phase, which we may asssume to be 1) is an Artin stack of finite type over $\mathbb{C}$ (cf. [65]). In general, it admits a coarse moduli space, which is a normal projective irreducible variety with $\mathbb{Q}$-factorial singularities. In the Picard rank one case, it was observed by Minamide, Yanagida, and Yoshioka (cf. [48]) that these moduli spaces are in fact isomorphic to moduli spaces of stable (twisted) sheaves on some Fourier–Mukai partner of S. This technique was generalized in [8, Sec. 7] by

Bayer and Macrì to arbitrary Picard rank. In particular, $M_\sigma(v)$ is a hyperkähler variety for v primitive and σ generic and every semistable object is stable (non-emptiness is essentially due to Kuleshov [38], Mukai [50], and Yoshioka, see e.g. [70] or [8, Cor. 6.9] in its final form). One might wonder what the precise relation between the moduli spaces $M_\sigma(v)$ and $M_\tau(v)$ for two generic stability conditions is. This was answered by Bayer and Macrì using a detailed analysis of the various types of walls (cf. [7, Thm. 5.7]):

Theorem 2.20. *Let S be a projective K3 surface, v primitive and $\sigma, \tau \in \mathrm{Stab}^\dagger(S)$ generic. Then $M_\sigma(v)$ and $M_\tau(v)$ are birational hyperkähler varieties.* □

As a corollary of Theorem 2.14 we obtain:

Corollary 2.21. *Let S be a projective K3 surface, v primitive and $\sigma, \tau \in \mathrm{Stab}^\dagger(S)$ generic stability conditions. There is an isomorphism of motives*

$$\mathfrak{h}(M_\sigma(v)) \simeq \mathfrak{h}(M_\tau(v)).$$ □

The following observation is an easy consequence:

Lemma 2.22. *Let S be a projective K3 surface, $v, w \in H^*_{\mathrm{alg}}(S, \mathbb{Z})$ primitive and σ a generic stability condition with respect to v. Assume that w is isotropic with $(v, w) = -r$. Then there exists a projective K3 surface S' and a Brauer class $\alpha \in \mathrm{Br}(S')$ such that the following holds:*

(1) $\mathfrak{h}(S) \simeq \mathfrak{h}(S')$,
(2) $\mathfrak{h}(M_\sigma(v))$ *is isomorphic to the motive of a moduli space of H'-Gieseker stable α-twisted sheaves of rank r on S', where H' is some polarization for S'.*

Proof. Let H be a generic polarization with respect to w and define $S' = M_H(w)$. Then S' is a K3 surface by results of Mukai (cf. [50]) and there exists a derived equivalence $\Phi\colon \mathrm{D}^{\mathrm{b}}(S) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(S', \alpha)$ where $\alpha \in \mathrm{Br}(S')$ is the obstruction to the existence of a universal family as explained in [20, Sec. 3.3]. It was recently observed by Huybrechts ([31, 34]) that derived equivalent (twisted) K3 surfaces have isomorphic Chow motives. Moreover, Φ induces an isomorphism $M_\sigma(v) \simeq M_{\Phi(\sigma)}(\Phi(v))$. Note that $\Phi(v)$ is a Mukai vector of rank r:

$$r = -(v, w) = -(\Phi(v), \Phi(w)) = -(\Phi(v), (0, 0, 1)) = \mathrm{rk}(\Phi(v)).$$

Passing to the large volume limit for some generic polarization H' with respect to w (cf. [16, Sec. 14]) we obtain a birational map which gives an isomorphism of Chow motives by Corollary 2.21. □

The above lemma allows one to reduce the rank of v in certain situations; in the case $r = 1$ we can now compare the moduli space with a Hilbert scheme of length n subschemes. Denote by $\beta(n)$ the set of partitions of n. Here, a partition λ of n is a nonincreasing sequence of positive integers $(\lambda_1, \ldots, \lambda_l)$ that sum to n and we call $l = l(\lambda)$ the length of λ. Note that $l(\lambda) = \sum_{i=1}^n a_i$

where $a_i = \#\{\lambda_j \mid \lambda_j = i\}$. We will use the shorthand $S^{(\lambda)} = S^{(a_1)} \times \ldots \times S^{(a_n)}$. Recall that the theory of (Chow) motives can easily be extended to quotients of smooth projective varieties by finite groups as was shown in [23], so in particular to symmetric products $S^{(k)}$.

Corollary 2.23. *Let S be a projective K3 surface, $v, w \in H^*_{\mathrm{alg}}(S, \mathbb{Z})$ be primitive and σ a generic stability condition with respect to v. Assume that $(v)^2 > -2$ and w is isotropic with $(v, w) = -1$. Let $\dim M_\sigma(v) = 2n$. There is an isomorphism of motives:*

$$\mathfrak{h}(M_\sigma(v)) \simeq \mathfrak{h}(S^{[n]}) \simeq \bigoplus_{\lambda \in \beta(n)} \mathfrak{h}(S^{(\lambda)}) \otimes \mathbb{L}^{n-l(\lambda)}.$$

In particular, $\mathfrak{h}(S)$ is finite dimensional if and only if $\mathfrak{h}(M_\sigma(v))$ is finite dimensional.

Proof. Let S' and H' be as above. Then $\mathfrak{h}(M_\sigma(v)) \simeq \mathfrak{h}(M_{H'}(u))$ where $u \in H^*_{\mathrm{alg}}(S', \mathbb{Z})$ has rank one; thus $M_{H'}(u)$ is isomorphic to the Hilbert scheme $S'^{[n]}$. The motive of a Hilbert scheme of a surface was computed by de Cataldo and Migliorini in [21], hence the formula above follows from the isomorphism $\mathfrak{h}(S) \simeq \mathfrak{h}(S')$. Note that $\mathfrak{h}(S^k) \simeq \mathfrak{h}(S^k, \mathrm{sym}_k, 0) \oplus \mathfrak{h}(S^k, [\Delta] - \mathrm{sym}_k, 0)$ and $\mathfrak{h}(S^{(k)}) = (S^k, \mathrm{sym}_k, 0)$. This proves one direction of the second claim because products resp. direct summands of finite dimensional motives are finite dimensional (Lemmata 2.12 and 2.13). The other direction follows similarly, since the summand for the partition $\lambda = (n)$ is $\mathfrak{h}(S \otimes \mathbb{L}^{n-1})$. $\square$

Remark 2.24. The corollary can also be deduced from the Torelli Theorem for hyperkähler varieties. Indeed, the lattice theoretic condition ensures that $M_\sigma(v)$ is birational to the Hilbert scheme $S'^{[n]}$ of some K3 surface S' (cf. [3, Prop. 5]). The birational map induces a Hodge isometry $\widetilde{H}(S, \mathbb{Z}) \simeq \widetilde{H}(S', \mathbb{Z})$ and thus a derived equivalence $\mathrm{D^b}(S) \simeq \mathrm{D^b}(S')$ by the derived Global Torelli Theorem (cf. [30, Ch. 10]). It follows as above that $\mathfrak{h}(M_\sigma(v)) \simeq \mathfrak{h}(S^{[n]})$.

Remark 2.25. Note that with the assumptions of the corollary, $M_\sigma(v)$ is in fact a fine moduli space (cf. [30, Proposition 10.24]). Indeed, if $w = (r', l', s')$ then $h = l' + (kr)A$ will be ample for $k \gg 0$ and A any ample class. Then $\gcd(r, (h.l), s) = 1$, which implies that the moduli space is fine.

We now come to the proof of Theorem 0.1. Let S be a projective (twisted) K3 surface (or an abelian surface). Assume that M is a smooth projective moduli space of stable (twisted) sheaves on S. See Remark 2.26 for comments on the case of a moduli space of σ-stable objects.

Proof of Theorem 0.1. Let E be a quasi-universal sheaf on $M \times S$ and F its transpose on $S \times M$. We use the following notation for the projections:

$$\begin{array}{ccccc} & & M \times S \times M & & \\ & \swarrow^{\pi_{12}} & \downarrow^{\pi} & \searrow^{\pi_{23}} & \\ M \times S & & M \times M & & S \times M \end{array}$$

and $\mathcal{E} = \pi_{12}^*(E)$, $\mathcal{F} = \pi_{23}^*(F)$ for the pullbacks. Consider the relative Ext sheaves $\mathcal{E}xt^i_\pi(\mathcal{E},\mathcal{F}) = \mathbf{R}^i(\pi_* \circ \mathcal{H}om)(\mathcal{E},\mathcal{F})$ and define

$$[\mathcal{E}xt^!_\pi] = \sum (-1)^i [\mathcal{E}xt^i_\pi(\mathcal{E},\mathcal{F})] \in \mathrm{K}(M \times M).$$

Note that in our case only $\mathcal{E}xt^1_\pi(\mathcal{E},\mathcal{F})$ and $\mathcal{E}xt^2_\pi(\mathcal{E},\mathcal{F})$ are non-zero. A computation of the Chern classes due to Markman ([47]) yields

$$c_m(-[\mathcal{E}xt^!_\pi]) = [\Delta_M] \in \mathrm{CH}^m(M \times M), \tag{1}$$

where m is the dimension of M (Lemma 4 of loc. cit. also applies to moduli spaces of stable twisted sheaves).

Consider the Chow groups $\mathrm{CH}^*(M \times M)_{\mathbb{Q}}$ as a unital ring with convolution of cycles and identity given by the diagonal. Define the following two-sided ideal generated by correspondences which factor through some power of S:

$$I = \langle \beta \circ \alpha \mid \alpha \in \mathrm{CH}^*(M \times S^k)_{\mathbb{Q}}, \beta \in \mathrm{CH}^*(S^k \times M)_{\mathbb{Q}}, k \geqslant 1 \rangle \subseteq \mathrm{CH}^*(M \times M)_{\mathbb{Q}}.$$

Note that I is closed under intersection products. Indeed, let $\alpha \in \mathrm{CH}^*(M \times S^k)_{\mathbb{Q}}$, $\beta \in \mathrm{CH}^*(S^k \times M)_{\mathbb{Q}}$, $\alpha' \in \mathrm{CH}^*(M \times S^{k'})_{\mathbb{Q}}$, $\beta' \in \mathrm{CH}^*(S^{k'} \times M)_{\mathbb{Q}}$ and denote by τ the involution of $M \times M \times M \times M$ interchanging the middle two factors:

$$\begin{aligned}(\beta \circ \alpha) \cdot (\beta' \circ \alpha') &= [{}^{\mathrm{t}}\Gamma_{\Delta_{M\times M}}]_*(\beta \circ \alpha \times \beta' \circ \alpha') = [{}^{\mathrm{t}}\Gamma_{\Delta_{M\times M}}]_* \circ \tau_*(\beta \times \beta' \circ \alpha \times \alpha') \\ &= [{}^{\mathrm{t}}\Gamma_{\tau\circ\Delta_{M\times M}}]_*(\beta \times \beta' \circ \alpha \times \alpha') = \left([{}^{\mathrm{t}}\Gamma_{\Delta_M}] \times [{}^{\mathrm{t}}\Gamma_{\Delta_M}]\right)_* (\beta \times \beta' \circ \alpha \times \alpha') \\ &= \left([{}^{\mathrm{t}}\Gamma_{\Delta_M}] \circ \beta \times \beta'\right) \circ \left(\alpha \times \alpha' \circ [\Gamma_{\Delta_M}]\right).\end{aligned}$$

The last equality follows from Lieberman's Lemma (cf. [52, Prop. 2.1.3]). We obtain a correspondence which factors through $S^{k+k'}$, so it is contained in I. We will conclude by showing that the class of the diagonal is contained in I.

A Grothendieck–Riemann–Roch computation gives:

$$\begin{aligned}\mathrm{ch}\Big(-[\mathcal{E}xt^!_\pi]\Big) &= -\mathrm{ch}\Big(\pi_![\mathbf{R}\mathcal{H}om(\mathcal{E},\mathcal{F})]\Big) = -\pi_*\Big(\mathrm{ch}[\mathbf{R}\mathcal{H}om(\mathcal{E},\mathcal{F})] \cdot \pi_2^*\mathrm{td}(S)\Big) \\ &= -\pi_*\Big(\pi_{12}^*\mathrm{ch}(E^\vee) \cdot \pi_{23}^*\mathrm{ch}(F) \cdot \pi_2^*\mathrm{td}(S)\Big),\end{aligned} \tag{2}$$

where $E^\vee$ denotes the derived dual of E and π_2 is the projection to S. Let $\alpha = \oplus\alpha^i = \mathrm{ch}(E^\vee) \cdot \pi_2^*\sqrt{\mathrm{td}(S)}$, $\beta = \oplus\beta^i = \mathrm{ch}(F) \cdot \pi_2^*\sqrt{\mathrm{td}(S)}$ and $n \in \mathbb{N}$. Considering only the codimension n part of (2) we find:

$$\mathrm{ch}_n\Big(-[\mathcal{E}xt^!_\pi]\Big) = -\sum_{i+j=n+2} \pi_*(\pi_{12}^*\alpha^i \cdot \pi_{23}^*\beta^j) \in I.$$

The codimension n part of the Chern character is given as a sum $\frac{(-1)^{n-1}}{(n-1)!}c_n + p$, where p is a polynomial in the Chern classes of degree less than n. Note that $c_1 = \mathrm{ch}_1$ is contained in I and therefore also $c_2 = \frac{1}{2}c_1^2 - \mathrm{ch}_2 \in I$. It follows iteratively that $c_n \in I$ for all n and therefore $[\Delta_M] \in I$ by (1). Thus, there are cycles $\gamma_i \in \mathrm{CH}^{e_i}(M \times S^{k_i})_{\mathbb{Q}}$, $\delta_i \in \mathrm{CH}^{d_i}(S^{k_i} \times M)_{\mathbb{Q}}$, for some $k_i \in \mathbb{N}$, such that

$$[\Delta_M] = \sum \delta_i \circ \gamma_i \in \mathrm{CH}^m(M \times M)_{\mathbb{Q}}. \tag{3}$$

Let $\delta = \bigoplus \delta_i$ viewed as a morphism of motives $\bigoplus \mathfrak{h}(S^{k_i})(n_i) \longrightarrow \mathfrak{h}(M)$ with $n_i = d_i - 2k_i$. Equation (3) asserts that $\gamma = \bigoplus \gamma_i$ defines a right inverse for δ, i.e. the following composition is the identity:

$$\mathfrak{h}(M) \xrightarrow{\ \gamma\ } \bigoplus \mathfrak{h}(S^{k_i})(n_i) \xrightarrow{\ \delta\ } \mathfrak{h}(M).$$

Hence, $\mathfrak{h}(M)$ is a direct summand of $\bigoplus \mathfrak{h}(S^{k_i})(n_i)$.

Moreover, we obtain a bound for the exponents k_i. Consider the filtration I_k of I generated by correspondences which factor through S^l with $l \leqslant k$. With the above notation we have $\mathrm{ch}_n \in I_1$ for all n and $I_k \cdot I_{k'} \subseteq I_{k+k'}$. Thus $k_i \leqslant \dim M$ for all i. □

Remark 2.26. The above argument also works for moduli spaces $M_\sigma(v)$ of σ-stable objects for a generic stability condition σ and primitive Mukai vector v. It was observed in [46] that Markman's computation of the Chern class can be carried out similarly in this case. Alternatively, one can use the fact that any such moduli space is birational to a moduli space of Gieseker stable sheaves and that birational hyperkähler varieties have isomorphic Chow motives.

Corollary 2.27. *Let S and M be as above. If $\mathfrak{h}(S)$ is finite dimensional, then $\mathfrak{h}(M)$ is finite dimensional as well.* □

Remark 2.28. We expect also that $\mathfrak{h}(S)$ is motivated by $\mathfrak{h}(M)$ (see the introduction). This holds for example in the case of a Hilbert scheme, and more generally for M as in Corollary 2.23. For fine moduli spaces it would follow from a conjecture of Addington [2]: A universal sheaf induces a Fourier–Mukai transform $F\colon \mathrm{D^b}(S) \longrightarrow \mathrm{D^b}(M)$ with right adjoint R. Addington conjectured that the composition of F and R splits as follows:

$$R \circ F \simeq \mathrm{id} \oplus \mathrm{id}[-2] \oplus \ldots \oplus \mathrm{id}[-2n+2].$$

If v and w are the Mukai vectors of the Fourier–Mukai kernels, we obtain:

$$[\Delta_S] = \frac{1}{n} v \circ w \in \mathrm{CH}^2(S \times S)_{\mathbb{Q}}.$$

It follows as above that $\mathfrak{h}(S)$ is a direct summand of $\bigoplus \mathfrak{h}(M)(n_i)$ for some $n_i \in \mathbb{Z}$.

2.5. **The Fano variety of lines.** We provide a short proof of Corollary 0.5. Let X be a cubic fourfold and F its Fano variety of lines. The Chow groups and motive of F were investigated in detail by Shen and Vial [63]. They studied Fourier transforms inducing a (particularly interesting) decomposition of the Chow ring, similar to the case of an abelian variety. The relation between the Chow groups of F and X given via the universal line (viewed as a correspondence) has been elucidated as well. We refrain from going into the details and recommend loc. cit. for further reading.

Proposition 2.29. *Let X be a cubic fourfold and F its Fano variety of lines. Then the transcendental motive $\mathfrak{t}(X)$ is a direct summand of $\mathfrak{h}(F)(-1)$. In particular, the motive of X is contained in* $\mathrm{Mot}(F)$.

Proof. The universal line $L \in \mathrm{CH}^3(F \times X)$ induces a morphism f of motives:

$$\mathfrak{h}(F)(-1) \xrightarrow{\ L\ } \mathfrak{h}(X) \xrightarrow{\ \pi_X^{4,\mathrm{tr}}\ } \mathfrak{t}(X).$$

Let K be any finitely generated field extension of $\mathbb{C}$. The only non-trivial rational Chow group of $\mathfrak{t}(X_K)$ is $\mathrm{CH}^3(\mathfrak{t}(X_K)) \simeq \mathrm{CH}_1(X_K)_{\mathrm{hom},\mathbb{Q}}$. Indeed, choose an embedding of K into the complex numbers and denote by Y the base change of X_K to $\mathbb{C}$, which is a smooth complex cubic fourfold. It is well known that the base change map $\mathrm{CH}^i(\mathfrak{t}(X_K)) \rightarrowtail \mathrm{CH}^i(\mathfrak{t}(Y))$ induced by a field extension is injective up to torsion (see e.g. [11, Lem. 1A.3]). Now use that $\mathrm{CH}^i(\mathfrak{t}(Y))$ vanishes for $i \neq 3$. The Chow group of one-cycles is universally generated by lines (cf. [62]) and the assertion thus follows from Lemma 2.9. □

3. Motives of special cubic fourfolds

In this section, we prove that there is an isomorphism of Chow motives $\mathfrak{t}(S)(1) \simeq \mathfrak{t}(X)$, if $X \in \mathcal{C}_d$ is a special cubic fourfold with an associated twisted K3 surface (S, α), i.e. d satisfies $(**')$. This generalizes work of Bolognesi, Pedrini [14], and Laterveer [43]. In [14], the authors obtained such an isomorphism in the case when $F(X) \simeq S^{[2]}$. Injectivity has been proven in [43] for cubic fourfolds invariant under a certain involution. Both cases are instances of Theorem 0.3 (see the comments in Subsection 3.1). We start with a well known fact:

Lemma 3.1. *Let S be a projective K3 surface and X a cubic fourfold. Then $\mathrm{CH}_0(S)_{\mathrm{hom}}$ and $\mathrm{CH}_1(X)_{\mathrm{hom}}$ are divisible and torsion-free.*

Proof. Divisibility of $\mathrm{CH}_0(S)_{\mathrm{hom}}$ follows easily by constructing a curve through any two given points and using the Jacobian of the normalization. The theorem of Rojtman [59] implies that this group is torsion-free. Let F be the Fano variety of lines in X. It is a hyperkähler variety, so its first Betti number vanishes and it follows as above that $\mathrm{CH}_0(F)_{\mathrm{hom}}$ is divisible and torsion-free. The universal line L induces a surjection

$$\mathrm{CH}_0(F)_{\mathrm{hom}} \longrightarrow \mathrm{CH}_1(X)_{\mathrm{hom}},$$

hence the assertion follows from the divisibility of $\operatorname{Ker}(L_*)$ which was proven by Shen and Vial ([63, Thm. 20.5, Lem. 20.6]). $\square$

Proof of Theorem 0.3. Since $\mathbb{C}$ is a universal domain, it suffices to prove the isomorphism on Chow groups. By a variant of Manin's identity principle (cf. [27, Lem. 1], [66, Lem. 3.2] or [56, Lem. 4.3]) this implies $\mathfrak{t}(S)(1) \simeq \mathfrak{t}(X)$. The results of Addington–Thomas [1] and Huybrechts [33] imply that there is an exact equivalence $\mathrm{D^b}(S) \simeq \mathcal{A}_X$ (resp. $\mathrm{D^b}(S,\alpha) \simeq \mathcal{A}_X$) if $X \in \mathcal{C}_d$ is generic and we consider this case first. Assume that $\alpha = 1$, i.e. d satisfies $(**)$. Consider the composition Φ of an exact equivalence $\mathrm{D^b}(S) \simeq \mathcal{A}_X$ and the inclusion $\mathcal{A}_X \subseteq \mathrm{D^b}(X)$. By [53], this functor is of Fourier–Mukai type, i.e. there is a complex $\mathcal{E} \in \mathrm{D^b}(S \times X)$, such that for all $\mathcal{G} \in \mathrm{D^b}(S)$:

$$\Phi(\mathcal{G}) \simeq p_*(\mathcal{E} \otimes q^*(\mathcal{G})),$$

where p and q are the projections. It follows that the left adjoint to Φ is of Fourier–Mukai type as well, say with kernel $\mathcal{F}$. Let $v = \mathrm{ch}(\mathcal{E}) \cdot \sqrt{\mathrm{td}(S \times X)}$ (resp. w) be the Mukai vector of $\mathcal{E}$ (resp. $\mathcal{F}$). It is an algebraic cycle with $\mathbb{Q}$-coefficients on $S \times X$ which needs not be of pure dimension. Denote by v^i (resp. w^i) its codimension i part. Since Φ is fully faithful, the convolution $w \circ v$ is rationally equivalent to the class of the diagonal $[\Delta_S]$ on $S \times S$. More precisely, the following equality holds in $\mathrm{CH}^2(S \times S)_{\mathbb{Q}}$:

$$[\Delta_S] = w^0 \circ v^6 + w^1 \circ v^5 + w^2 \circ v^4 + w^3 \circ v^3 + w^4 \circ v^2 + w^5 \circ v^1 + w^6 \circ v^0. \tag{4}$$

Recall that the cohomologically trivial part of the Chow groups of S and X are concentrated in codimension two resp. three. The induced action of v on Chow groups is compatible with the action on cohomology. Thus, $w^3 \circ v^3$ is the only summand on the right hand side of (4) acting non-trivially on $\mathrm{CH}_0(S)_{\mathrm{hom},\mathbb{Q}}$, i.e. the following composition is the identity:

$$\mathrm{CH}_0(S)_{\mathrm{hom},\mathbb{Q}} \xrightarrow{v^3_*} \mathrm{CH}_1(X)_{\mathrm{hom},\mathbb{Q}} \xrightarrow{w^3_*} \mathrm{CH}_0(S)_{\mathrm{hom},\mathbb{Q}}.$$

This proves injectivity of v^3_*. For the surjectivity consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{D^b}(S) & \xrightarrow{\sim} & \mathcal{A}_X & \hookrightarrow & \mathrm{D^b}(X) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{K}(S)_{\mathbb{Q}} & \xrightarrow{\sim} & \mathrm{K}(\mathcal{A}_X)_{\mathbb{Q}} & \longrightarrow & \mathrm{K}(X)_{\mathbb{Q}} \\
\downarrow v & & & \searrow^{\phi} & \downarrow v \\
\mathrm{CH}^*(S)_{\mathbb{Q}} & & \xrightarrow{v_*} & & \mathrm{CH}^*(X)_{\mathbb{Q}} \\
\uparrow & & & & \uparrow \\
\mathrm{CH}_0(S)_{\mathrm{hom},\mathbb{Q}} & & \xrightarrow{v^3_*} & & \mathrm{CH}_1(X)_{\mathrm{hom},\mathbb{Q}}.
\end{array}$$

Commutativity of the middle diagram follows from the Grothendieck–Riemann–Roch Theorem. It suffices to show that the image of $\phi\colon \mathrm{K}(\mathcal{A}_X)_{\mathbb{Q}} \longrightarrow \mathrm{CH}^*(X)_{\mathbb{Q}}$ contains $\mathrm{CH}_1(X)_{\mathrm{hom},\mathbb{Q}}$. Indeed, this would imply that any $\beta \in \mathrm{CH}_1(X)_{\mathrm{hom},\mathbb{Q}}$ lifts to some $\alpha \in \mathrm{CH}^*(S)_{\mathbb{Q}}$ such that $v_*(\alpha) = \beta$. Since the action of v on cohomology is injective, α is homologically trivial, i.e. $\alpha \in \mathrm{CH}_0(S)_{\mathrm{hom},\mathbb{Q}}$.

Recall that by a result of Paranjape [55] (see also [61, Cor. 4.3]) $\mathrm{CH}_1(X)$ is generated by lines. Let $i\colon \ell \subseteq X$ be the inclusion of a line and consider the associated second syzygy sheaf $\mathcal{F}_\ell$ of $\mathcal{I}_\ell(1)$, defined by:

$$0 \longrightarrow \mathcal{F}_\ell \longrightarrow H^0(X, \mathcal{I}_\ell(1)) \otimes \mathcal{O}_X \xrightarrow{\mathrm{ev}} \mathcal{I}_\ell(1) \longrightarrow 0.$$

Here, $\mathcal{O}_X(1)$ is the induced polarization of $X \subseteq \mathbb{P}^5$ and ev is the evaluation map which is surjective (cf. [39, Lem. 5.1]). A straightforward computation in loc. cit. shows that $\mathcal{F}_\ell$ is contained in $\mathcal{A}_X$. Next, we compute the Mukai vector of $\mathcal{F}_\ell$:

$$v(\mathcal{F}_\ell) = v(\mathcal{O}_X^{\oplus 4}) - v(\mathcal{I}_\ell(1)) = v(\mathcal{O}_X^{\oplus 4}) - v(\mathcal{O}_X(1)) + v(\mathcal{O}_\ell(1)).$$

Using the Grothendieck–Riemann–Roch Theorem one finds:

$$\begin{aligned} v(\mathcal{O}_\ell(1)) &= \mathrm{ch}(\mathcal{O}_\ell) \cdot \mathrm{ch}(\mathcal{O}_X(1)) \cdot \mathrm{td}(X)^{\frac{1}{2}} = i_*(\mathrm{td}(\ell)) \cdot \mathrm{ch}(\mathcal{O}_X(1)) \cdot \mathrm{td}(X)^{-\frac{1}{2}} \\ &= ([\ell] + [\mathrm{pt}]) \cdot \mathrm{ch}(\mathcal{O}_X(1)) \cdot \mathrm{td}(X)^{-\frac{1}{2}}, \end{aligned}$$

where $[\mathrm{pt}] \in \mathrm{CH}_0(X) \simeq \mathbb{Z}$ is the class of any closed point (X is rationally connected). The Todd class of X is a polynomial in the class of a hyperplane section $h = c_1(\mathcal{O}_X(1))$, in fact

$$\mathrm{td}(X) = 1 + \frac{3}{2}h + \frac{5}{4}h^2 + \frac{3}{4}h^3 + \frac{1}{3}h^4.$$

Therefore, $v(\mathcal{O}_\ell(1)) = [\ell] + \frac{5}{4}[\mathrm{pt}]$ and

$$\begin{aligned} \phi([\mathcal{F}_\ell] - [\mathcal{F}_{\ell'}]) &= v(\mathcal{O}_X^{\oplus 4}) - v(\mathcal{O}_X(1)) + v(\mathcal{O}_\ell(1)) - (v(\mathcal{O}_X^{\oplus 4}) - v(\mathcal{O}_X(1)) + v(\mathcal{O}_{\ell'}(1))) \\ &= v(\mathcal{O}_\ell(1)) - v(\mathcal{O}_{\ell'}(1)) = [\ell] - [\ell'], \end{aligned}$$

for each pair of lines ℓ and ℓ', which proves surjectivity of ϕ since $\mathrm{CH}_1(X)_{\mathrm{hom},\mathbb{Q}}$ is generated by cycles of this form.

So far, we proved that $Z = v^3$ induces an isomorphism $\mathrm{CH}_0(S)_{\mathrm{hom},\mathbb{Q}} \xrightarrow{\sim} \mathrm{CH}_1(X)_{\mathrm{hom},\mathbb{Q}}$. As mentioned earlier, a variant of Manin's identity principle gives that Z also induces an isomorphism of motives $\mathfrak{t}(S)(1) \simeq \mathfrak{t}(X)$, which extends to an isomorphism $\mathfrak{h}(S)(1) \simeq \mathbb{L} \oplus \mathfrak{h}^{\mathrm{pr}}(X) \oplus \mathbb{L}^3$. Indeed, the Picard rank ρ of S equals $\rho_2 - 1$ with $\rho_2 = \dim H^{2,2}(X, \mathbb{Q})$. Thus, there are cycles W, $W' \in \mathrm{CH}^3(S \times X)_{\mathbb{Q}}$ such that

$$^{\mathrm{t}}W' \circ W = [\Delta_S], \quad W \circ {}^{\mathrm{t}}W' = \pi_X^2 + \pi_X^{\mathrm{pr}} + \pi_X^4. \tag{5}$$

This will be useful for the specialization argument below.

Next, assume that d satisfies $(**')$, i.e. $\mathrm{D^b}(S,\alpha) \simeq \mathcal{A}_X$. The composition with the inclusion is again of Fourier–Mukai type (cf. [17]) and the formalism of Mukai vectors works in the twisted case as well (see [36] for details). For $E \in \mathrm{Coh}(S \times X, \alpha^{-1} \boxtimes 1)$ locally free and $n = \mathrm{ord}(\alpha)$ the order of the Brauer class, $E^{\otimes n}$ is naturally an untwisted sheaf and one defines (cf. [34, Sec. 2.1]):

$$v(E) = \sqrt[n]{\mathrm{ch}(E^{\otimes n})} \cdot \sqrt{\mathrm{td}(S \times X)}.$$

The n-th root can be obtained formally, since $\mathrm{rk}(E) \neq 0$. Using a locally free resolution, this definition extends to twisted coherent sheaves. Define the cycle Z as above. The proof now works analogously, replacing $\mathrm{D^b}(S)$ by $\mathrm{D^b}(S,\alpha)$ and $\mathrm{K}(S)$ by $\mathrm{K}(S,\alpha)$.

Finally, we prove the assertion for any $X_0 \in \mathcal{C}_d$ via specialization. Let $T \subseteq \mathcal{C}_d$ be a curve passing through the point corresponding to X_0 such that there are families of K3 surfaces (resp. cubic fourfolds) $\mathcal{S}$ and $\mathcal{X}$ over T with an exact equivalence $\mathrm{D^b}(\mathcal{S}_s) \simeq \mathcal{A}_{\mathcal{X}_s}$ over a very general point $s \in T$ and $\mathcal{X}_0 \simeq X_0$ for a closed point $0 \in T$ (see [1]). Write S_0 for the fibre of $\mathcal{S}$ over 0.

By a standard argument (see e.g. [60, Lem. 8]), the very general fibre of a smooth, proper morphism of complex varieties specializes to the central fibre (the very general fibre is isomorphic as an abstract variety to the geometric generic fibre since $\mathbb{C}$ is a universal domain). Applying this to the families $\mathcal{S} \times_T \mathcal{X}$ and $\mathcal{S} \times_T \mathcal{S}$ we may assume that T is the spectrum of a complete discrete valuation ring $R \simeq \mathbb{C}[\![t]\!]$ with generic point η and closed point 0. Write $K = \mathbb{C}(\!(t)\!)$ for its fraction field and $\bar{K}$ for an algebraic closure of K.

Let W, $W' \in \mathrm{CH}^3(\mathcal{S}_{\bar\eta} \times_{\bar K} \mathcal{X}_{\bar\eta})$ be as above, such that (5) holds. In fact, all cycles of (5) are defined over a finite extension $\mathbb{C}(\!(t^{\frac{1}{n}})\!)$ of K. Replacing R by $\mathbb{C}[\![t^{\frac{1}{n}}]\!]$, we may assume that the cycles W and W' are defined over K. Recall the specialization map for Chow groups (see [25, Ch. 10.1] for details), which is compatible with intersection product, pullback and proper pushforward. We obtain cycles W_0, $W_0' \in \mathrm{CH}^3(S_0 \times X_0)_{\mathbb{Q}}$ such that equalities of the form (5) hold. Thus, W_0 induces an isomorphism of motives $\mathfrak{h}(S_0)(1) \simeq \mathbb{L} \oplus \mathfrak{h}^{\mathrm{pr}}(X_0) \oplus \mathbb{L}^3$. The action on Chow groups restricts to an isomorphism of homologically trivial cycles $\mathrm{CH}_0(S)_{\mathrm{hom},\mathbb{Q}} \xrightarrow{\sim} \mathrm{CH}_1(X)_{\mathrm{hom},\mathbb{Q}}$ induced by $\pi^{4,\mathrm{tr}}_{X_0} \circ W_0 \circ \pi^{2,\mathrm{tr}}_{S_0}$. In fact, $\mathrm{CH}_0(S)_{\mathrm{hom}}$ and $\mathrm{CH}_1(X)_{\mathrm{hom}}$ are both divisible and torsion-free (see Lemma 3.1), hence tensoring with $\mathbb{Q}$ is a bijection and we obtain an isomorphism of integral Chow groups. □

Corollary 3.2. *Let $X \in \mathcal{C}_d$ be a special cubic fourfold with d satisfying $(**')$ and S an associated (twisted) K3 surface. Then $\mathfrak{h}(X)$ is finite dimensional if and only if $\mathfrak{h}(S)$ is finite dimensional. Assume moreover that $\dim H^{2,2}(X,\mathbb{Z}) \geqslant 20$. Then $\mathfrak{h}(X)$ is finite dimensional.*

Proof. The above theorem evidently implies $\mathfrak{h}(X) \simeq \mathbb{1} \oplus \mathfrak{h}(S)(1) \oplus \mathbb{L}^2 \oplus \mathbb{L}^4$ and we conclude using Proposition 2.16. □

3.1. **Examples.** We include a comparison with the work of Bolognesi, Pedrini [14] and review some concrete geometric constructions producing the K3 surface.

Example 3.3 (Cubic fourfolds containing a plane)**.** Consider the divisor $\mathcal{C}_8 \subseteq \mathcal{C}$. It corresponds exactly to the cubic fourfolds X containing a plane (cf. [68, Sec. 3]). In this case, there is the following standard construction: Let $\widetilde{X}$ be the blow-up of X along a plane P. Projecting X from P onto a disjoint plane in $\mathbb{P}^5$ yields a rational map which can be resolved to give a morphism $q\colon \widetilde{X} \longrightarrow \mathbb{P}^2$. The fiber of q over a point $x \in \mathbb{P}^2$ is the residual surface of the intersection $\overline{xP} \cap X$. Generically, it is a smooth quadric surface, i.e. isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and has two different rulings. The discriminant divisor of q is a sextic curve in $\mathbb{P}^2$ over which each fibre is singular with only one ruling. More precisely, let $F(\widetilde{X}/\mathbb{P}^2)$ be the relative Fano variety of lines with universal line $L \subseteq F(\widetilde{X}/\mathbb{P}^2) \times \widetilde{X}$. The Stein factorization of the projection $L \longrightarrow \mathbb{P}^2$ gives a diagram:

$$\begin{array}{ccccc} L & \longrightarrow & \widetilde{X} & \xrightarrow{\;q\;} & X \\ \downarrow & & & & \\ S & & & & \\ \downarrow & & & & \\ \mathbb{P}^2, & & & & \end{array}$$

where $S \longrightarrow \mathbb{P}^2$ is a double cover, branched along a sextic curve, which is smooth for a general choice of X. Thus, S is a K3 surface. The projection $L \longrightarrow S$ is a $\mathbb{P}^1$ bundle (a Brauer–Severi variety) and induces a Brauer class $\alpha \in \mathrm{Br}(S)$. Kuznetsov showed that there is an exact equivalence $\mathrm{D^b}(S, \alpha) \simeq \mathcal{A}_X$ (cf. [40, Thm. 4.3]). Note that $d = 8$ indeed satisfies condition $(**')$; therefore $\mathfrak{t}(S)(1) \simeq \mathfrak{t}(X)$ by Theorem 0.3.

It is well known that rationality of the cubic fourfold X follows, if q has a rational section. This holds true if there is an additional surface $W \subseteq X$ such that $\deg(W) - \langle P, W\rangle$ is odd. In this case, it was observed in [14, Sec. 8] that the isomorphism $\mathfrak{t}(S)(1) \simeq \mathfrak{t}(X)$ would follow from finite dimensionality of $\mathfrak{h}(S)$.

Example 3.4 (Cubic fourfolds with an automorphism of order three)**.** Let X be a cubic fourfold given by an equation of the form

$$f(x_0, x_1, x_2) + g(x_3, x_4, x_5) = 0,$$

where f and g are homogeneous polynomials of degree three. Then X is invariant under the automorphism σ of $\mathbb{P}^5$ given by

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \longmapsto [x_0 : x_1 : x_2 : e^{\frac{2\pi i}{3}} x_3 : e^{\frac{2\pi i}{3}} x_4 : e^{\frac{2\pi i}{3}} x_5].$$

Thus, there is an induced automorphism σ_F of the Fano variety $F(X)$, which is in fact symplectic, i.e. $\sigma_F|_{H^{2,0}} = \mathrm{id}$ (see e.g. [24] for a classification of polarized symplectic automorphisms of $F(X)$). Consider the cubic surfaces $Z_1 = \{f(x_0, x_1, x_2) - s^3 = 0\}$ and $Z_2 = \{g(x_3, x_4, x_5) - t^3\}$ in $\mathbb{P}^3$

with s resp. t as additional variables. The rational map

$$([x_0 : x_1 : x_2 : s], [x_3 : x_4 : x_5 : t]) \longmapsto [\frac{x_0}{s} : \frac{x_1}{s} : \frac{x_2}{s} : \frac{x_3}{t} : \frac{x_4}{t} : \frac{x_5}{t}]$$

induces a degree three morphism $\widetilde{Z_1 \times Z_2} \longrightarrow X$ from the blow-up of $Z_1 \times Z_2$ along $E_1 \times E_2$, where E_i is the cubic curve in Z_i defined by the vanishing of s resp. t (see e.g. [19, Prop. 1.2]). Note that finite dimensionality of $\mathfrak{h}(X)$ follows from Proposition 2.11 and Lemma 2.13 since rational surfaces have finite dimensional motives. Moreover, this morphism can be used to find two disjoint planes P_1 and P_2 contained in X; if $\ell_i \subseteq Z_i$ are lines (recall that Z_i contains 27 of them) the image of the product $\ell_1 \times \ell_2$ is a plane in X and certain choices of lines produce disjoint planes (cf. [19, Rem. 2.4]). There is a birational map from $P_1 \times P_2$ to X sending a pair of points (x, y) to the residual point of the intersection $\overline{xy} \cap X$. The indeterminacy locus $S \subseteq P_1 \times P_2$ parametrizes lines contained in X joining the two planes. It is a complete intersection of divisors of type $(1, 2)$ and $(2, 1)$, i.e. S is a K3 surface (see [26, Ex. 5.9]). Resolving the indeterminacy locus gives an isomorphism $\mathrm{Bl}_S(P_1 \times P_2) \xrightarrow{\sim} \mathrm{Bl}_{P_1 \cup P_2}(X)$ which induces $\mathfrak{t}(S)(1) \simeq \mathfrak{t}(X)$ by comparing homologically trivial cycles. In fact, the cubic fourfold X satisfies condition $({*}{*}{*})$ since the Fano variety of X is birational to the Hilbert scheme $S^{[2]}$ (see loc. cit.).

Example 3.5 (Cubic fourfolds with an involution)**.** Consider the involution σ on $\mathbb{P}^5$ given by

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \longmapsto [x_0 : x_1 : x_2 : x_3 : -x_4 : -x_5].$$

A cubic X invariant under σ is always of the form

$$\{F(x_0, x_1, x_2, x_3) + x_4^2 L_1 + x_5^2 L_2 + x_4 x_5 L_3 = 0\},$$

where F is homogeneous of degree three and the L_i are linear forms in $x_0, \ldots, x_3$. Note that the fixed locus of σ in $\mathbb{P}^5$ is the union of $\mathbb{P}^3 = \{x_4 = x_5 = 0\}$ and the line $\ell = \{[0 : 0 : 0 : 0 : x_4 : x_5]\}$. Thus, the fixed locus in X consists of a cubic surface W and the line ℓ.

It was shown in [24] that σ induces a symplectic involution on the Fano variety $F(X)$. Moreover, the fixed locus in $F(X)$ can be described explicitly. It consists of the line ℓ, the 27 lines contained in W and a K3 surface S. The surface S parametrizes lines contained in X joining W and ℓ. It is a double cover of the cubic W branched along the degree 6 curve $L_3^2 - L_1 L_2$. This suggests that S is associated to X: The inclusion $S \subseteq F(X)$ induces an isomorphism $H^{2,0}(F(X)) \simeq H^{2,0}(S)$ and an isomorphism of transcendental lattices. Composing with the incidence correspondence, we get $T(S)(-1) \simeq T(X)$. It is not apparent that this is an isometry. An isomorphism $\mathfrak{t}(S)(1) \simeq \mathfrak{t}(X)$ was nevertheless established by Bolognesi and Pedrini (cf. [14, Sec. 5.2]) building on work of Laterveer [43].

Example 3.6 (Cyclic cubic fourfolds)**.** Let $f(x_0, \ldots, x_4)$ be a homogeneous polynomial of degree three, defining a smooth cubic threefold $C \subseteq \mathbb{P}^4$. A cyclic cubic fourfold is a triple cover $X \longrightarrow \mathbb{P}^4$

ramified along C. It is a smooth cubic hypersurface $X \subseteq \mathbb{P}^5$ with an equation:

$$f(x_0, \ldots, x_4) + x_5^3 = 0$$

and covering automorphism $\sigma\colon X \xrightarrow{\sim} X$ given by:

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \longmapsto [x_0 : x_1 : x_2 : x_3 : x_4 : e^{\frac{2\pi i}{3}} x_5].$$

It was shown in [42] that the motive of a cyclic cubic fourfold X is finite dimensional. If X satisfies condition $(**')$ and S is an associated (twisted) K3 surface, then $\mathfrak{t}(S)(1) \simeq \mathfrak{t}(X)$ and $\mathfrak{h}(S)$ is finite dimensional as well. Unfortunately, it is not clear which K3 surfaces can be associated to some X as above. Note that the family of cyclic cubic fourfolds contains the Fermat cubic, so in particular it has non-trivial intersection with the divisor $\mathcal{C}_8$ of cubic fourfolds containing a plane. However, there exists an example of a cyclic Pfaffian cubic fourfold containing no plane (see [13, Prop. 5.1]).

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