Fulton's trace formula for coherent cohomology

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Einleitung

In der vorliegenden Arbeit studieren wir die Anzahl rationaler Punkte auf Schemata über endlichen Körpern mittels Kohomologie kohärenter Garben. Wir geben den Beweis einer Spurformel an, die auf Fulton zurückgeht. Die Idee ist hierbei ähnlich zu dem was Weil vorschlug, nämlich dass die rationalen Punkte als Fixpunkte des Frobenius Endomorphismus erfasst werden können. Im Jahr 1978 bewies Fulton einen Fixpunktsatz über $\mathbb{F}_q$, aus dem sich die folgende Identität ableitet:

$$|X(\mathbb{F}_q)| \equiv \sum_{i=0}^{d} (-1)^i \operatorname{trace}(\mathrm{Frob_q} \,| H^i(X, \mathcal{O}_X)) \mod p.$$

Hierbei ist X ein eigentliches Schema über $\mathbb{F}_q$ und d ist seine Dimension.
In seinem Beweis konstruiert er eine Grothendieck Gruppe für Schemata über endlichen Körpern, die den Frobenius mit einbezieht. Diese Gruppen sind sogar Vektorräume über $\mathbb{F}_q$ und verhalten sich somit recht unterschiedlich zu gewöhnlichen Grothendieck Gruppen von kohärenten Garben oder Vektorbündeln. Es stellt sich heraus, dass die Dimension dieses Vektorraums der Anzahl rationaler Punkte entspricht.

Diese Bachelorarbeit zielt auf eine einfach zugängliche Präsentation der Methoden ab, die Fulton verwendet. Anschließend diskutieren wir Beispiele und Anwendungen der Spurformel. Wir werden zunächst verschiedene Begriffe einführen und die erwähnte Konstruktion im Detail besprechen, bevor wir durch vorbereitende Resultate in Richtung des Beweises der Formel manövrieren.

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Contents

1 Introduction

In this thesis, we study the number of rational points on schemes over finite fields, using cohomology of coherent sheaves. More specifically, we will present a proof of a trace formula which is due to Fulton. The idea is similar to what Weil proposed, namely that the rational points can be detected as fixed points of the Frobenius endomorphism.
In 1978, Fulton proved a fixed point formula, taking place in $\mathbb{F}_q$, from which the following identity is deduced:

$$|X(\mathbb{F}_q)| \equiv \sum_{i=0}^{d}(-1)^i \operatorname{trace}(\mathrm{Frob_q}|H^i(X,\mathcal{O}_X)) \mod p.$$

Here, X is a proper scheme over $\mathbb{F}_q$ and d denotes its dimension.
In his proof, he constructs a Grothendieck group for schemes over finite fields, that involves the Frobenius. These groups are actually vector spaces over $\mathbb{F}_q$ and, therefore, behave quite differently to the usual Grothendieck groups of coherent sheaves or vector bundles. It turns out that the dimension of this vector space equals the number of rational points.

This thesis aims at a clear presentation of the methods used by Fulton, followed by examples and applications. We introduce several notions and describe the construction in detail before we maneuver through some preliminary results towards the proof of the formula.

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Notations and Conventions: A *variety* will be an integral, separated scheme of finite type over a field and a *curve* is a variety of dimension 1. By $(Sch/\mathbb{F}_q)$ we mean the category of proper schemes X over $\mathbb{F}_q$. Let $\mathcal{E}$, $\mathcal{F}$ be $\mathcal{O}_X$-modules, then we write $\mathscr{H}om(\mathcal{E},\mathcal{F})$ for the sheaf hom and $\mathcal{E}^* = \mathscr{H}om(\mathcal{E},\mathcal{O}_X)$ for the dual of $\mathcal{E}$. If $\mathcal{F}$ is coherent, we denote the dimensions of the cohomology groups, which are finite-dimensional vector spaces (cf. [21]), by $h^i(\mathcal{F}) = \dim_{\mathbb{F}_q} H^i(X,\mathcal{F})$. Furthermore, we let $X(\mathbb{F}_{q^m})$ be the (finite) set of rational points over $\mathbb{F}_{q^m}$, equipped with the reduced scheme structure. Its cardinality is $|X(\mathbb{F}_{q^m})|$.
The subscripts in $\mathbb{P}^n_{\mathbb{F}_q}$ and $\mathcal{O}_{\mathbb{P}^n}(l)$ will frequently be omitted.

2 F-modules

The notion of an F-module will be introduced and illustrated with some easy examples. This section follows primarily [18, §5].

2.1 The Frobenius morphism

A scheme of characteristic p always comes with a certain endomorphism, the absolute Frobenius. We investigate properties of this morphism and then focus on schemes that are *Frobenius split.*

Definition 2.1. *Let X be a scheme over $\mathbb{F}_q$. The unique endomorphism* $\mathrm{Frob_q}$ *of X which is the identity on topological spaces and raises each regular function to its q^{th} power, is called the (absolute) Frobenius morphism. On structure sheaves, it is given by*

$$\begin{aligned} \mathcal{O}_X &\to \mathrm{Frob_{q*}}(\mathcal{O}_X) \\ a &\mapsto a^q. \end{aligned}$$

Remark 2.2. If $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $\mathcal{F}$ and $\mathrm{Frob_{q*}}(\mathcal{F})$ are isomorphic as sheaves of abelian groups and there is a natural isomorphism $H^i(X,\mathcal{F}) \simeq H^i(X,\mathrm{Frob_{q*}}(\mathcal{F}))$. We will make use of this implicitly.

Lemma 2.3. *Let X be a scheme of finite type over $\mathbb{F}_q$. Then the Frobenius is a finite morphism of schemes.*

Proof. If C is a ring, $\mathrm{Frob_{q*}}(C)$ denotes the C-module structure given by the Frobenius on C.
Now locally, X is the affine variety $\operatorname{Spec} A$ with A of finite type over $\mathbb{F}_q$. First, assume that $A = \mathbb{F}_q[x_1,\dots,x_m]$. Then $\mathrm{Frob_{q*}}(A)$ is obviously a finite module over A.
For general A, consider a surjection $B = \mathbb{F}_q[x_1,\dots,x_m] \twoheadrightarrow A$. By the first case, for some $N > 0$ we obtain a commutative diagram:

$$\begin{array}{ccc} B^{\oplus N} & \twoheadrightarrow & A^{\oplus N} \\ \downarrow\!\!\downarrow & & \vdots\ \exists \\ \mathrm{Frob_{q*}}(B) & \twoheadrightarrow & \mathrm{Frob_{q*}}(A). \end{array}$$

Then the dotted arrow is surjective as well. □

Lemma 2.4. *Let X be a scheme of finite type over $\mathbb{F}_q$. Then X is regular if and only if* $\mathrm{Frob_q}\colon X \to X$ *is a flat morphism.*

Proof. A proof can be found in [15, Thm. 2.1, p. 773]. □

Remark 2.5. Since $\mathbb{F}_q$ is a perfect field, the previous lemma can also be stated as 'X is smooth if and only if $\mathrm{Frob_{q*}}(\mathcal{O}_X)$ is a vector bundle'.

We state the following lemma without a proof.

Lemma 2.6. *Let $f\colon X \to Y$ be a proper morphism of Noetherian schemes, with Y reduced and connected, and $\mathcal{E}$ a coherent sheaf on X, flat over Y. Then for all integers $p \geq 0$ the following conditions are equivalent:*

(i) $y \mapsto \dim H^p(X_y, \mathcal{E}_y)$ *is a constant function,*

(ii) $R^p f_*(\mathcal{E})$ *is a locally free sheaf on* Y *and, for all* $y \in Y$, *the natural map*

$$R^p f_*(\mathcal{E}) \otimes k(y) \to H^p(X_y, \mathcal{E}_y)$$

is an isomorphism.

Proof. A proof can be found in [17, II §5 Cor. 2, pp. 50–51]. □

Proposition 2.7. *If* X *is a smooth curve over* $\mathbb{F}_q$, *then* $\mathrm{Frob}_{q*}(\mathcal{O}_X)$ *is a vector bundle of rank* q.

Proof. Let K be the function field of X and $K^{1/q}$ the field extension induced by the Frobenius. If we write $k := \mathbb{F}_q$ then K is a finite extension of $k(t)$ and $[k(t)^{1/q} : k(t)] = q$. Then by multiplicativity of degrees:

$$[K^{1/q} : K] \cdot [K : k(t)] = [K^{1/q} : k(t)^{1/q}] \cdot [k(t)^{1/q} : k(t)].$$

Since $[K : k(t)] = [K^{1/q} : k(t)^{1/q}]$, we find that $\deg(\mathrm{Frob}_q) = [K^{1/q} : K] = q$. By Lemma 2.4, the Frobenius is a flat morphism. Thus for all $y \in Y$ it holds that (cf. [16, Ex. 1.25, p. 176]):

$$\dim_{k(y)} H^0(X_y, \mathcal{O}_{X_y}) = \deg(\mathrm{Frob}_q) = q.$$

We use Lemma 2.6 the conclude. □

Remark 2.8. We briefly mention that the pullback of line bundles under the Frobenius is easy to compute. If $\mathcal{L} \in \mathrm{Pic}(X)$ for X a scheme over $\mathbb{F}_q$, we use a cocycle description $\{\varphi_{i,j}\} \in H^1(X, \mathcal{O}_X^*)$. The image under $H^1(X, \mathcal{O}_X^*) \xrightarrow{\mathrm{Frob}_q} H^1(X, \mathcal{O}_X^*)$ is $\{\varphi_{i,j}^q\}$, which is a cocycle description of $\mathcal{L}^q$, i.e. $\mathrm{Frob}_q^*(\mathcal{L}) \simeq \mathcal{L}^q$.

In contrast to that, it is not obvious what the pushforward of sheaves under the Frobenius yields. Even for line bundles this is a priori not clear.

For the structure sheaf, we treat the case of $\mathbb{P}^1$ in the following example, but refer to the literature (e.g. [20, Ex. 8.10., 8.12.]) for projective spaces of higher dimensions.

Example 2.9. The following observation is due to Schwede [20, Ex. 3.4].
We compute $\mathrm{Frob}_{q*}(\mathcal{O}_{\mathbb{P}^1})$, using that $\mathbb{P}^1$ is smooth and that every locally free sheaf of finite rank on $\mathbb{P}^1$ decomposes as a direct sum of line bundles (see e.g. [13]).
The rank of $\mathrm{Frob}_{q*}(\mathcal{O}_{\mathbb{P}^1})$ is q, so we can write $\mathrm{Frob}_{q*}(\mathcal{O}_{\mathbb{P}^1}) = \mathcal{O}(l_1) \oplus \ldots \oplus \mathcal{O}(l_q)$, with $l_i \in \mathbb{Z}$. Since

$$1 = h^0(\mathcal{O}_{\mathbb{P}^1}) = h^0(\mathrm{Frob}_{q*}(\mathcal{O}_{\mathbb{P}^1})) = \sum_{i=1}^{q} h^0(\mathcal{O}(l_i)),$$

we may assume that $l_1 = 0$ and $l_i < 0$ for $i \geq 2$. Furthermore, by the projection formula:

$$q + 1 = h^0(\mathrm{Frob}_{q*}(\mathcal{O}(q))) = h^0(\mathrm{Frob}_{q*}(\mathcal{O}_{\mathbb{P}^1})(1)) = \sum_{i=1}^{q} h^0(\mathcal{O}(l_i + 1)),$$

and we deduce that

$$h^0(\mathcal{O}(l_2 + 1)) + \ldots + h^0(\mathcal{O}(l_q + 1)) = q - 1.$$

Then neccessarily $h^0(\mathcal{O}(l_2{+}1)) = \ldots = h^0(\mathcal{O}(l_q{+}1)) = 1$ as $l_i < 0$ for $i \geq 2$. Hence $l_2 = \ldots = l_q = -1$ and consequently $\mathrm{Frob}_{q*}(\mathcal{O}_{\mathbb{P}^1}) = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}(-1) \oplus \ldots \oplus \mathcal{O}(-1)$.

If X is a variety, so in particular X is reduced, then the Frobenius is injective as a morphism of sheaves and one could ask whether this has a split.

Definition 2.10. *A scheme X over $\mathbb{F}_q$ is called Frobenius split (or simply split), if the Frobenius $\mathcal{O}_X \to \mathrm{Frob}_{q*}(\mathcal{O}_X)$ splits as a morphism of $\mathcal{O}_X$-modules.*

Lemma 2.11. *Every regular, affine variety $\operatorname{Spec} A$ of finite type over $\mathbb{F}_q$ is Frobenius split.*

Proof. We follow the proof in [20, Prop. 3.3, p. 3].

Consider the map $\mathrm{ev}\colon \mathrm{Hom}(\mathrm{Frob}_{q*}(A), A) \to A$ defined by evaluation at 1. Then the Frobenius on $X = \operatorname{Spec} A$ splits if and only if ev is surjective. Observe that surjectivity of ev is a local property since $\mathrm{Frob}_{q*}(A)$ is a finite A-module. Therefore, it would suffice to prove that for every point $x \in X$ the Frobenius on $\operatorname{Spec} \mathcal{O}_{X,x}$ splits, so we may assume that A is local.

By Lemma 2.4, $\mathrm{Frob}_{q*}(A)$ is flat, hence free over A and projection onto one component defines a surjective map $f\colon \mathrm{Frob}_{q*}(A) \to A$. Let $f(a) = 1$ for some $a \in \mathrm{Frob}_{q*}(A)$. Then $g(-) := f(a \cdot -)$ is a split of the Frobenius. □

It turns out that most of the varieties are not split, but those which are, admit remarkable geometric and cohomological properties. A nice presentation for criteria and consequences of Frobenius splitting can be found in [3].

Lemma 2.12. *Let X be a smooth projective variety of dimension n over $\mathbb{F}_q$ and ω_X its canonical bundle. Then X is Frobenius split if and only if the natural map $H^n(X, \omega_X) \to H^n(X, \omega_X^q)$ is injective (or equivalently nonzero, since $H^n(X, \omega_X)$ is 1-dimensional).*

Proof. We follow the proof in [19, Prop. 9].

Let $\mathcal{C}$ be the cokernel of the Frobenius $\mathcal{O}_X \to \mathrm{Frob}_{q*}(\mathcal{O}_X)$. Then C is a locally free sheaf since X is regular and so the Frobenius locally splits by the previous lemma. Then the exact sequence

$$0 \to \mathcal{O}_X \to \mathrm{Frob}_{q*}(\mathcal{O}_X) \to \mathcal{C} \to 0$$

is split if and only if the dual sequence

$$0 \to \mathcal{C}^* \to \mathrm{Frob}_{q*}(\mathcal{O}_X)^* \to \mathcal{O}_X \to 0$$

is split. The latter is split if and only if the identity on $\mathcal{O}_X$ can be lifted to $\mathrm{Frob}_{q*}(\mathcal{O}_X)^*$. A morphism $\mathcal{O}_X \to \mathrm{Frob}_{q*}(\mathcal{O}_X)^*$ is a global section of $\mathrm{Frob}_{q*}(\mathcal{O}_X)^*$, so this is equivalent to

$$H^0(X, \mathrm{Frob}_{q*}(\mathcal{O}_X)^*) \to H^0(X, \mathcal{O}_X)$$

being surjective. By Serre duality, the dual of the last map is

$$H^n(X, \omega_X) \to H^n(X, \mathrm{Frob}_{q*}(\omega_X^q)) \simeq H^n(X, \omega_X^q),$$

where we made use of the projection formula and Remark 2.8. □

2.2 The category of F-modules

We define a category $\mathrm{Coh_F}(X)$ of coherent sheaves, equipped with a Frobenius action, which will be the key ingredient of the discussion in the following sections. It turns out, that this category is in fact abelian.

Definition 2.13. *An abelian category is an additive category $\mathcal{A}$ such that:*

(i) *Every morphism in $\mathcal{A}$ has a kernel and cokernel,*

(ii) *every monic in $\mathcal{A}$ is the kernel of its cokernel and*

(iii) *every epi in $\mathcal{A}$ is the cokernel of its kernel.*

Definition 2.14. *Let X be a scheme of finite type over $\mathbb{F}_q$ and $\mathcal{M}$ a coherent $\mathcal{O}_X$-module. Then $\mathcal{M}$ is called a coherent Frobenius module, or F-module for short, if $\mathcal{M}$ admits a Frobenius action, i.e. a morphism of sheaves of $\mathcal{O}_X$-modules $F_{\mathcal{M}}: \mathcal{M} \to \mathrm{Frob_{q*}}(\mathcal{M})$.*

The set of pairs $(\mathcal{M}, F_{\mathcal{M}})$ forms a category $\mathrm{Coh_F}(X)$. A morphism $(\mathcal{M}, F_{\mathcal{M}}) \to (\mathcal{N}, F_{\mathcal{N}})$ is a morphism $\mathcal{M} \to \mathcal{N}$ that commutes with the Frobenius actions.

Remark 2.15. The pushforward $\mathrm{Frob_{q*}}(\mathcal{M})$ admits another $\mathcal{O}_X$-structure than $\mathcal{M}$, namely the one given by $\mathcal{O}_X \to \mathrm{Frob_{q*}}(\mathcal{O}_X)$. With this structure, one has that $F_{\mathcal{M}}(am) = a^q F_{\mathcal{M}}(m)$ for any local sections a of $\mathcal{O}_X$ and m of $\mathcal{M}$.

According to Remark 2.2, there are induced $\mathbb{F}_q$-linear maps on cohomology, which we will call $F_{\mathcal{M}}$ as well. Each rational point $x \in X(\mathbb{F}_q)$ creates an $\mathbb{F}_q$-linear endomorphism $F_{\mathcal{M}}(x) \in \mathrm{End}_{\mathbb{F}_q}(\mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} k(x))$ of the fibre of $\mathcal{M}$, whose traces will be of interest later on.

Lemma 2.16. *Let X be a scheme of finite type over $\mathbb{F}_q$. Then $\mathrm{Coh_F}(X)$ is an abelian category.*

Proof. Most of the requirements are checked easily, so we will concentrate on the existence of kernels and cokernels. For a given $\mathcal{M} \to \mathcal{N}$ we use the forgetful functor $\mathrm{Coh_F}(X) \to \mathrm{Coh}(X)$ to define $\mathcal{K}$, $\mathcal{C}$ as the kernel resp. cokernel in $\mathrm{Coh}(X)$, which is an abelian category (see e.g. [11, II Prop. 5.7]). By commutativity of

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \mathcal{N} \\ \downarrow & & \downarrow \\ \mathrm{Frob_{q*}}(\mathcal{M}) & \longrightarrow & \mathrm{Frob_{q*}}(\mathcal{N}) \end{array}$$

there are induced Frobenius actions on $\mathcal{K}$ and $\mathcal{C}$ so we are done. □

It is very convenient to make these generalizations, because the trace formula will eventually follow by considering the F-module $(\mathcal{O}_X, \mathrm{Frob_q})$.

2.3 F-modules on $\mathbb{P}^n$

In the following subsection we investigate F-modules on the projective space.

Example 2.17. An F-module $(\mathcal{M}, F_{\mathcal{M}})$ on $\mathbb{P}^n$ corresponds to a graded module over $S = \mathbb{F}_q[x_0, \dots, x_n]$ (cf. [11, II 5.15]) with a certain endomorphism, that we will call a graded Frobenius action. Indeed, using the projection formula and Remark 2.8, for every i there is an induced morphism

$$\mathcal{M}(i) \to \mathrm{Frob}_{q*}(\mathcal{M}) \otimes \mathcal{O}(i) \to \mathrm{Frob}_{q*}(\mathcal{M}(qi)),$$

which gives rise to an endomorphism φ of the graded S-module $M = \Gamma_*(\mathcal{M}) := \bigoplus_{i \geq 0} \Gamma(\mathbb{P}^n, \mathcal{M}(i))$, satisfying $\varphi(M_i) \subseteq M_{qi}$ and $\varphi(au) = a^q \varphi(u)$ for $a \in S$ and $u \in M$.
If conversely M is a graded S-module and φ an endomorphism with the above properties, we define a Frobenius action on $\widetilde{M}$ locally via $F_{\widetilde{M}}(\frac{u}{x_i^N}) = \frac{\varphi(u)}{x_i^{qN}}$ for every $u \in M_N$.
Given a coherent F-module $(\mathcal{M}, F_{\mathcal{M}})$ and $M = \Gamma_*(\mathcal{M})$, with φ as described above, we have an isomorphism of F-modules $\mathcal{M} \simeq \widetilde{M}$. We will make use of this description in the proof of the Localization Theorem in Section 4.

Lemma 2.18. *Let $l \in \mathbb{Z}$ and $\mathcal{O}(-l) \in \mathrm{Pic}(\mathbb{P}^n)$. Then there is a 1:1 correspondence*

$$\{\varphi \text{ Frobenius action on } \mathcal{O}(-l)\} \xleftrightarrow{1:1} H^0(\mathbb{P}^n, \mathcal{O}(ql - l)).$$

If we view the set of Frobenius actions as an $\mathbb{F}_q$-vector space, this is actually an isomorphism. In particular, if $\mathcal{L}$ is an ample line bundle on $\mathbb{P}^n$, it does not admit any nonzero Frobenius action.

Proof. We make use of Remark 2.8 and the identification $\mathscr{H}\!om(\mathcal{E}, \mathcal{F}) \simeq \mathcal{E}^* \otimes \mathcal{F}$ for $\mathcal{E}$ locally free of finite rank and $\mathcal{F}$ a sheaf of modules (cf. [11, II Ex. 5.1.]):

$$\begin{aligned}\mathrm{Hom}(\mathcal{O}(-l), \mathrm{Frob}_{q*}\, \mathcal{O}(-l)) &\simeq \mathrm{Hom}(\mathrm{Frob}_q^*\, \mathcal{O}(-l), \mathcal{O}(-l)) \\ &\simeq \mathrm{Hom}(\mathcal{O}(-ql), \mathcal{O}(-l)) \\ &\simeq H^0(\mathbb{P}^n, \mathcal{O}(ql - l)).\end{aligned}$$

Explicitly, fix a homogeneous polynomial $h \in H^0(\mathbb{P}^n, \mathcal{O}(ql-l))$. We obtain a graded Frobenius action φ on $S(-l)$ by $\varphi(g) = g^q \cdot h$ and an induced Frobenius action on $\mathcal{O}(-l)$, as described above. □

Example 2.19. We apply the previous lemma to the polynomials $x_i^q f$ resp. $x_i f$ with $f \in H^0(\mathbb{P}^n, \mathcal{O}(ql - l - q))$ and obtain graded Frobenius actions φ on $S(-l)$ resp. ψ on $S(-l+1)$. Multiplication with x_i defines a morphism of graded S-modules $S(-l) \xrightarrow{\cdot x_i} S(-l+1)$ and we check that this commutes with φ and ψ:

$$\varphi(g) \cdot x_i = g^q \cdot x_i^q f \cdot x_i = g^q \cdot x_i^{q+1} f = (g x_i)^q \cdot x_i f = \psi(g \cdot x_i).$$

Thus, the associated morphism $\mathcal{O}(-l) \xrightarrow{\cdot x_i} \mathcal{O}(-l+1)$ of line bundles on $\mathbb{P}^n$ is a morphism of F-modules.

Example 2.20. Note that for any scheme X the induced Frobenius action on the cotangent sheaf Ω_X vanishes. If locally f is a regular function, the morphism $\mathrm{Frob}_q^*(\Omega_X) \to \Omega_X$ maps df to $df^q = 0$, which vanishes due to the Leibniz rule and the characteristic p dividing q. Thus, the adjoint morphism is zero as well. Fortunately, on the projective space we can use the Euler sequence to define a non-trivial Frobenius action on $\Omega_{\mathbb{P}^n}$.

We consider the line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$ together with the Frobenius action given by x_i^{q-1}. Then the right hand square of the following diagram commutes and we get an induced action on the kernel $\Omega_{\mathbb{P}^n}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_{\mathbb{P}^n} & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} & \longrightarrow & \mathcal{O}_{\mathbb{P}^n} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Frob}_{q*}(\Omega_{\mathbb{P}^n}) & \longrightarrow & \mathrm{Frob}_{q*}\left(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1}\right) & \longrightarrow & \mathrm{Frob}_{q*}(\mathcal{O}_{\mathbb{P}^n}) & \longrightarrow & 0. \end{array}$$

On $\mathbb{P}^1$ for instance, this induces the action given by $(x_0x_1)^{q-1}$ on $\mathcal{O}_{\mathbb{P}^1}(-2) = \Omega_{\mathbb{P}^1}$.

3 The Grothendieck group of F-modules

In 1957, Grothendieck introduced the idea of assigning to each vector bundle over a smooth variety X an invariant, its class in a group. Subject to a relation for each short exact sequence of vector bundles, the set of all these classes forms an abelian group $K(X)$. This notion was later generalized to any exact category. We will make use of this in a slightly adjusted form.

3.1 Modified Grothendieck groups

The prototype of the following construction is described in Example 3.4.

Construction 3.1. Let $\mathcal{A}$ be an exact category with $F \in \mathrm{End}(\mathcal{A})$ an endofunctor. Consider the category $\mathcal{A}_F$ of all pairs (A, φ) with $A \in \mathrm{ob}(\mathcal{A})$ and $A \xrightarrow{\varphi} F(A)$. A morphism $(A,\varphi) \xrightarrow{f} (B,\psi)$ is a commutative diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ F(A) & \xrightarrow{F(f)} & F(B). \end{array}$$

Dually, we define $\mathcal{A}^F$ = category of all pairs (A,φ) with $F(A) \xrightarrow{\varphi} A$.
Then $\mathcal{A}_F$, $\mathcal{A}^F$ are easily checked to be exact again and one can consider their Grothendieck groups.
We divide out by the further relation $(A, \varphi_1 + \varphi_2) \sim (A,\varphi_1) + (A,\varphi_2)$ to get

$$K_F(\mathcal{A}) := K(\mathcal{A}_F)/\sim \quad \text{and} \quad K^F(\mathcal{A}) := K(\mathcal{A}^F)/\sim .$$

The class of (A,φ) will be denoted by $[A,\varphi]$ or simply $[A]$ if the morphism is understood. We observe that under some circumstances these constructions are functorial.

If $\mathcal{A}$ and $\mathcal{B}$ are exact, with endofunctors F resp. G, and $\mathcal{A} \xrightarrow{\gamma} \mathcal{B}$ is exact and compatible with F and G, i.e. $\gamma \circ F \simeq G \circ \gamma$, then there are induced morphisms

$$K_F(\mathcal{A}) \xrightarrow{\gamma} K_G(\mathcal{B}) \quad \text{and} \quad K^F(\mathcal{A}) \xrightarrow{\gamma} K^G(\mathcal{B}).$$

In fact, for abelian categories $\mathcal{A}$ with exact endofunctors, we already get induced morphisms for left exact functors γ whose derived functors are bounded in the sense that $R^j\gamma(A) = 0$ for all $A \in \mathcal{A}$ and j big enough:

$$R^i\gamma \circ F \simeq R^i(\gamma \circ F) \simeq R^i(G \circ \gamma) \simeq G \circ R^i\gamma.$$

So, if $(A,\varphi) \in \mathcal{A}_F$ then $R^i\gamma(A)$ admits an induced morphism $R^i\gamma(\varphi)$ and we define

$$\gamma[A,\varphi] := \sum_{i\geq 0}(-1)^i[R^i\gamma(A), R^i\gamma(\varphi)].$$

Of course, we assumed that $\mathcal{A}$ has enough injectives. An analogue definition is possible if γ is only right exact.

Remark 3.2. If $\mathcal{A}$ is exact and L is a left adjoint to R, then $K^L(\mathcal{A}) \simeq K_R(\mathcal{A})$.

Remark 3.3. One could, of course, consider usual K-theory of $\mathcal{A}$ and try to relate this to the above construction. For $F = \mathrm{id}$ there is an exact functor $\mathcal{A} \to \mathcal{A}_{\mathrm{id}}$ taking A to (A,id) and a split $\mathcal{A}_{\mathrm{id}} \to \mathcal{A}$ taking (A,φ) to A. This induces

$$K(\mathcal{A}) \hookrightarrow K(\mathcal{A}_{\mathrm{id}}) \twoheadrightarrow K_{\mathrm{id}}(\mathcal{A})$$

and one could ask whether the composition is still injective. The answer will be 'No' in general, by the following example.

Example 3.4. Let X be a scheme over $\operatorname{Spec} R$ and f any endomorphism of X. Let $\mathcal{A} = \mathrm{Coh}(X)$ be the category of coherent sheaves and $F = f_*$. A pair $(\mathcal{M},\varphi)$ is a morphism $\mathcal{M} \xrightarrow{\varphi} f_*(\mathcal{M})$ resp. $f_*(\mathcal{M}) \xrightarrow{\varphi} \mathcal{M}$ of $\mathcal{O}_X$-modules. Then $K_F(\mathcal{A})$ and $K^F(\mathcal{A})$ are actually R-modules via $r\cdot[\mathcal{M},\varphi] := [\mathcal{M}, r\cdot\varphi]$.

If $X = \mathbb{P}^n$ is the projective space over a field k of characteristic p, it is a well known fact that $K(\mathrm{Coh}(\mathbb{P}^n)) \simeq \mathbb{Z}^{\oplus n+1}$ (see e.g. [26, Ex. 6.14]). This is a free abelian group and contains no p-torsion, while $K_{\mathrm{id}}(\mathrm{Coh}(\mathbb{P}^n))$ is a vector space over k. Consequently, the morphism from the previous example can not be injective.

3.2 $K_\bullet$ and $K^\bullet$ of F-modules

We will apply the construction to the category $\mathrm{Coh}_{\mathrm{F}}(X)$ and obtain a certain $\mathbb{F}_q$-vector space. Its most important property is, that every additive function (in a sense that will be made precise) $\mathrm{Coh}_{\mathrm{F}}(X) \to \mathbb{F}_q$ factors through it.

Definition 3.5. *Let X be a scheme of finite type over $\mathbb{F}_q$ and $F := \mathrm{Frob_q}$ the Frobenius. The abelian group*

$$K_\bullet(X) := K_{F_*}(\mathrm{Coh}(X))$$

is called the Grothendieck group of F-modules. In fact, this is an $\mathbb{F}_q$-vector space by Example 3.4.

Caution: In this thesis, $K_\bullet$ denotes the modified Grothendieck group described in Construction 3.1. This is not the same as usual K-theory of schemes, which we will not consider at all.

To be more explicit, $K_\bullet(X)$ is the quotient of the free abelian group generated by all isomorphism classes of F-modules $(\mathcal{M}, F_{\mathcal{M}})$ subject to the following relations:

(A) $(\mathcal{M},F_{\mathcal{M}}) = (\mathcal{M}',F_{\mathcal{M}'}) + (\mathcal{M}'',F_{\mathcal{M}''})$ for every short exact sequence

$$0 \to (\mathcal{M}',F_{\mathcal{M}'}) \to (\mathcal{M},F_{\mathcal{M}}) \to (\mathcal{M}'',F_{\mathcal{M}''}) \to 0,$$

(B) $(\mathcal{M}, F_1 + F_2) = (\mathcal{M}, F_1) + (\mathcal{M}, F_2)$ for all morphisms of $\mathcal{O}_X$-modules $F_1, F_2 \colon \mathcal{M} \to \mathrm{Frob}_{q*}(\mathcal{M})$, where $\mathcal{M}$ is a coherent sheaf on X.

A function $l \colon \mathrm{Coh}_{\mathrm{F}}(X) \to A$ with A an abelian group will be called *additive*, if it factors through $K_\bullet(X)$. The Grothendieck group is universal, as it represents the functor that takes A to the set of additive functions with values in A.

Remark 3.6. The relation (B) is very important; it implies that the class of an F-module $(\mathcal{M}, 0)$ with the zero action is trivial, since $[\mathcal{M}, 0] = [\mathcal{M}, 0] + [\mathcal{M}, 0]$. Later on, we will see that the class already vanishes if the Frobenius action is nilpotent.

We now like to compare the Grothendieck group of F-modules with a very similar construction, using vector bundles instead of coherent sheaves.

Definition 3.7. *Let X be a scheme of finite type over $\mathbb{F}_q$ and denote by* $\mathrm{Loc}(X)$ *the exact category of locally free sheaves of finite rank (cf. [6, p. 270]). The Frobenius $F = \mathrm{Frob}_q$ induces an exact endofunctor F^* and we can form the category* $\mathrm{Loc}(X)^{F^*}$. *Define*

$$K^\bullet(X) \coloneqq K^{F^*}(\mathrm{Loc}(X)).$$

Due to adjointness of F^* and F_*, a morphism $F^*(\mathcal{E}) \to \mathcal{E}$ with $\mathcal{E}$ locally free, corresponds to a Frobenius action on $\mathcal{E}$.
Furthermore, we observe that $K^\bullet(X)$ is actually a commutative ring, using the tensor product of $\mathcal{O}_X$-modules. If $\mathcal{E}$, $\mathcal{G}$ are locally free with actions φ, ψ we let

$$[\mathcal{E}, \varphi] \cdot [\mathcal{G}, \psi] \coloneqq [\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{G}, \varphi \otimes \psi].$$

This is well defined, since locally free sheaves are flat. As a consequence, every morphism $f \colon X \to Y$ induces a ringhomomorphism $f^* \colon K^\bullet(Y) \to K^\bullet(X)$ by pulling back locally free sheaves, together with their Frobenius actions. Again, using the tensor product, this turns $K_\bullet(X)$ into a $K^\bullet(X)$-module.
Since every locally free sheaf of finite rank is coherent, there is a morphism

$$c_X \colon K^\bullet(X) \to K_\bullet(X)$$

of $K^\bullet(X)$-modules, which we will call the 'Poincaré homomorphism' (following Fulton and Lang [8, p. 165]).

3.3 Basic properties of $K_\bullet$

We compute the Grothendieck group of $\mathrm{Spec}\,\mathbb{F}_q$ and study properties of $K_\bullet$ that follow immediately from the relations (A) and (B).

Lemma 3.8. *Let $X = \mathrm{Spec}\,\mathbb{F}_q$ and $x \in X$ the unique point. Then there is an isomorphism of $\mathbb{F}_q$-vector spaces:*

$$K_\bullet(X) \simeq \mathbb{F}_q, \quad [\mathcal{M}, F_{\mathcal{M}}] \mapsto \mathrm{trace}(F_{\mathcal{M}}(x)).$$

Proof. We follow the proof in [18, Lemma 5.5 p. 42].

A coherent sheaf on X is simply a finite-dimensional $\mathbb{F}_q$-vector space. The Frobenius on X is the identity, so $(V, f) \in \mathrm{Coh}_{\mathrm{F}}(X)$ is a vector space together with a linear endomorphism f. The trace defines an additive function and induces a morphism $K_\bullet(X) \to \mathbb{F}_q$. This morphism has a section $\mathbb{F}_q \xrightarrow{s} K_\bullet(X)$

taking $\lambda \in \mathbb{F}_q$ to $[\mathbb{F}_q, \lambda \cdot \mathrm{id}]$, so it suffices to show that s is surjective.
Assume that $\dim(V) \geq 2$. If V has a proper, nonzero f-invariant subspace W, there is a short exact sequence

$$0 \to (W, f) \to (V, f) \to (V/W, \bar{f}) \to 0$$

and we reduce to W resp. V/W which have both dimension strictly smaller than $\dim(V)$. Otherwise pick a basis v_i of V and let $f(v_i) = w_i$. Let

$$f_1(v_i) = \begin{cases} v_1 & i = 1, \\ w_i - v_i & i \geq 2 \end{cases} \text{ and } f_2(v_i) = \begin{cases} w_1 - v_1 & i = 1, \\ v_i & i \geq 2 \end{cases}.$$

Then $f = f_1 + f_2$ and f_1, f_2 have invariant nonzero subspaces, so we can make use of the crucial relation (B) to reduce to the first case. By induction, we may assume that $\dim(V) = 1$. Then the endomorphism is given by $\lambda \in \mathbb{F}_q$. □

Lemma 3.9. *Let X be a scheme of finite type over $\mathbb{F}_q$ and $[\mathcal{M}, F_\mathcal{M}] \in K_\bullet(X)$, then*

(i) $[\mathcal{M}, F_\mathcal{M}] = 0$ *if* $F_\mathcal{M}^m = 0$ *for some* $m > 0$.

(ii) $[\mathcal{M}, F_\mathcal{M}] = \sum_{i=1}^{k} [\mathcal{M}, F_{i,i}]$ *if* $\mathcal{M} = \mathcal{M}_1 \oplus \ldots \oplus \mathcal{M}_k$ *and* $F_{i,j}$ *is the composition* $\mathcal{M}_i \to \mathcal{M} \xrightarrow{F_\mathcal{M}} \mathrm{Frob_{q*}}(\mathcal{M}) \to \mathrm{Frob_{q*}}(\mathcal{M}_j)$.

(iii) $\sum_{i=0}^{n} (-1)^i [\mathcal{F}_i] = \sum_{i=0}^{n} (-1)^i [\mathcal{H}_i]$ *if*

$$0 \to \mathcal{F}_n \to \ldots \to \mathcal{F}_0 \to 0$$

is a complex of F-modules and $\mathcal{H}_i$ denotes the i^{th} homology sheaf.

(iv) $[\mathcal{M}] = \sum_{i=0}^{N} (-1)^i [\mathcal{G}_i]$ *if there is a finite resolution of F-modules:*

$$0 \to \mathcal{G}_N \to \ldots \to \mathcal{G}_0 \to \mathcal{M} \to 0.$$

Proof. We follow the proof in [18, Lemma 5.11/ 5.12, pp. 44–45].
By $F_\mathcal{M}^m$ we mean the composition

$$\mathcal{M} \to \mathrm{Frob_{q*}}(\mathcal{M}) \to \ldots \to \mathrm{Frob^m_{q*}}(\mathcal{M})$$

and we say that the action $F_\mathcal{M}$ is nilpotent if this composition vanishes for some $m > 0$.
The class of the kernel $(\mathrm{Ker}(F_\mathcal{M}), 0)$ in the Grothendieck group is zero by Remark 3.6. Furthermore, the induced Frobenius action $\overline{F}_\mathcal{M}$ on $\mathcal{M}/\mathrm{Ker}(F_\mathcal{M})$ is nilpotent, namely $\overline{F}_\mathcal{M}^{m-1} = 0$. Using the short exact sequence

$$0 \to (\mathrm{Ker}(F_\mathcal{M}), 0) \to (\mathcal{M}, F_\mathcal{M}) \to (\mathcal{M}/\mathrm{Ker}(F_\mathcal{M}), \overline{F}_\mathcal{M}) \to 0,$$

the assertion follows by induction on m.
For (ii), let $G_{i,j}$ be the composition $\mathcal{M} \to \mathcal{M}_i \xrightarrow{F_{i,j}} \mathrm{Frob_{q*}}(\mathcal{M}_j) \to \mathrm{Frob_{q*}}(\mathcal{M})$. Then $F_\mathcal{M} = \sum_{i,j} G_{i,j}$ and, therefore, $[\mathcal{M}, F_\mathcal{M}] = \sum_{i,j} [\mathcal{M}, G_{i,j}]$. But $G_{i,j}^2 = 0$ for

$i \neq j$, hence $[\mathcal{M}, F_{\mathcal{M}}] = \sum_{i=1}^{k} [\mathcal{M}, G_{i,i}]$. To conclude, we use the short exact sequence

$$0 \to (\mathcal{M}_i, F_{i,i}) \to (\mathcal{M}, G_{i,i}) \to (\bigoplus_{j \neq i} \mathcal{M}_j, 0) \to 0.$$

In order to prove (iii), we note that the sheaves $\mathcal{K}_i = \operatorname{Ker}(\mathcal{F}_i \to \mathcal{F}_{i-1})$, $\mathcal{J}_i = \operatorname{Im}(\mathcal{F}_i \to \mathcal{F}_{i-1})$, $\mathcal{H}_i = \mathcal{K}_i / \mathcal{J}_{i+1}$ are F-modules with the induced actions and it holds that

$$[\mathcal{F}_i] = [\mathcal{K}_i] + [\mathcal{J}_i] \text{ and } [\mathcal{H}_i] = [\mathcal{K}_i] - [\mathcal{J}_{i+1}].$$

Therefore

$$\begin{aligned}\sum_{i=0}^{n} (-1)^i [\mathcal{F}_i] &= \sum_{i=0}^{n} (-1)^i ([\mathcal{K}_i] + [\mathcal{J}_i]) = \sum_{i=0}^{n} (-1)^i ([\mathcal{H}_i] + [\mathcal{J}_{i+1}] + [\mathcal{J}_i]) \\ &= \left(\sum_{i=0}^{n} (-1)^i [\mathcal{H}_i] \right) + [\mathcal{J}_0] + [\mathcal{J}_{n+1}] = \sum_{i=0}^{n} (-1)^i [\mathcal{H}_i].\end{aligned}$$

Of course, (iii) implies (iv) as all homology sheaves $\mathcal{H}_i$ vanish. □

3.4 Functorial properties of $K_\bullet$

We observe that $K_\bullet \colon (Sch/\mathbb{F}_q) \to \operatorname{Vect}(\mathbb{F}_q)$ is indeed a covariant functor:

Proposition 3.10. *For a proper morphism $f \colon X \to Y$ we obtain an induced map f_* on Grothendieck groups via*

$$f_*([\mathcal{M}]) := \sum_{i \geq 0} (-1)^i [R^i f_*(\mathcal{M})] \in K_\bullet(Y).$$

If $g \colon Y \to Z$ is another proper morphism, then we have $(g \circ f)_* = g_* \circ f_* \colon K_\bullet(X) \to K_\bullet(Z)$.

Proof. For a coherent $\mathcal{O}_X$-module $\mathcal{M}$ and a proper morphism, the higher direct images are coherent as well (cf. [10, EGA III 3.2.1]). It remains to show that all $R^i f_*(\mathcal{M})$ admit Frobenius actions and that the above formula is compatible with the relations (A) and (B). We already know that there are induced actions on all $R^i f_*(\mathcal{M})$ as pointed out in Construction 3.1, since the Frobenius commutes with any morphism of schemes, but let us examine this in more detail.

If $(\mathcal{M}, F_{\mathcal{M}}) \in \operatorname{Coh}_{\mathrm{F}}(X)$, then restricting to some $f^{-1}(U)$ for $U \subseteq Y$ an affine open subset, gives a Frobenius action on $\mathcal{M}|_{f^{-1}(U)}$ and thus an induced morphism on $\Gamma(U, R^i f_*(\mathcal{M})) = H^i(f^{-1}(U), \mathcal{M})$. These glue to give the desired one on $R^i f_*(\mathcal{M})$. To see that we have successfully defined an additive function $\operatorname{Coh}_{\mathrm{F}}(X) \to K_\bullet(Y)$ we consider a short exact sequence of F-modules,

$$0 \to (\mathcal{M}', F_{\mathcal{M}'}) \to (\mathcal{M}, F_{\mathcal{M}}) \to (\mathcal{M}'', F_{\mathcal{M}''}) \to 0.$$

The derived functors $R^i f_*$ form a cohomological δ-functor (see e.g. [25, 2.4.6] for a proof of the dual statement for left derived functors), so there is an induced long exact sequence. The definition of δ-functors already implies that this is compatible with the Frobenius actions:

$$\ldots \to R^i f_*(\mathcal{M}') \to R^i f_*(\mathcal{M}) \to R^i f_*(\mathcal{M}'') \to R^{i+1} f_*(\mathcal{M}') \to \ldots .$$

As cohomology vanishes in large dimensions (see [11, II Thm. 2.7]), we get a well defined relation for the alternating sums

$$\sum_{i\geq 0}(-1)^i[R^if_*(\mathcal{M})] = \sum_{i\geq 0}(-1)^i[R^if_*(\mathcal{M}')] + \sum_{i\geq 0}(-1)^i[R^if_*(\mathcal{M}'')] \in K_\bullet(Y).$$

Each of the functions $R^if_*\colon \mathrm{Coh}_{\mathrm{F}}(X) \to K_\bullet(Y)$ factors through the type (B) relations, so this is also true for their alternating sum.
For the second part of the proposition, we make use of a Grothendieck spectral sequence (see e.g. [25, 5.8.3] or [9, 2.4.1]) for the left exact functors f_* and g_*: There exists a cohomological (first quadrant) spectral sequence

$$E_2^{p,q} = R^pg_*(R^qf_*(\mathcal{M})) \implies H^{p+q} = R^{p+q}(g\circ f)_*(\mathcal{M}).$$

Let us temporarily define $\sum E_k := \sum_{p,q}(-1)^{p+q}[E_k^{p,q}] \in K_\bullet(Z)$, for $k \geq 2$. This is well defined, as our spectral sequence is bounded in the sense that $E_k^{p,q} = 0$ for p or $q > \max\{\dim(X), \dim(Y)\}$.
By some homological yoga one sees that $\sum E_k = \sum E_{k+1}$. Moreover, by convergence of the spectral sequence, for some k big enough there are short exact sequences of the form

$$0 \longrightarrow F^l \longrightarrow F^{l-1} \longrightarrow E_k^{l-1,n-l+1} \longrightarrow 0,$$

for all $1 \leq l \leq n$. Here $0 \subseteq F^n \subseteq \ldots \subseteq F^0 = H^n$ is a filtration of H^n. Using $F^n = E_k^{n,0}$ we conclude that $[R^n(g\circ f)_*(\mathcal{M})] = \sum_{p+q=n}[E_k^{p,q}]$. Then by definition

$$(g\circ f)_*[\mathcal{M}] = \sum_{n\geq 0}(-1)^n[R^n(g\circ f)_*(\mathcal{M})] = \sum_{n\geq 0}(-1)^n\sum_{p+q=n}[E_k^{p,q}] = \sum E_k.$$

Finally, we compute

$$\begin{aligned} g_*(f_*[\mathcal{M}]) &= g_*(\sum_{q\geq 0}(-1)^q[R^qf_*(\mathcal{M})]) \\ &= \sum_{q\geq 0}(-1)^q\sum_{p\geq 0}(-1)^p[R^pg_*(R^qf_*(\mathcal{M}))] \\ &= \sum_{p,q}(-1)^{p+q}[R^pg_*(R^qf_*(\mathcal{M}))] = \sum E_2, \end{aligned}$$

which finishes the proof. □

We discuss a remarkable functorial property of $K_\bullet$, which fails for usual K-theory of schemes.

Lemma 3.11. *Let $i\colon Z \to X$ be a closed immersion. Then the restriction defines a morphism $i^*\colon K_\bullet(X) \to K_\bullet(Z)$ and $i_*\colon K_\bullet(Z) \to K_\bullet(X)$ is a split. As a consequence, $K_\bullet(Z)$ is a direct summand of $K_\bullet(X)$.*

Proof. We follow the proof in [7, Lemma 2 p. 192].

We will implicitly use that i_* is fully faithful and talk about morphisms of sheaves on Z but in fact use their pushforwards to X. Now, if $\mathcal{I}$ denotes the ideal sheaf of Z in X, we note that $F_{\mathcal{M}}(\mathcal{I}\mathcal{M}) \subseteq \mathcal{I}^q\mathcal{M}$ and we obtain an induced action $\overline{F}_{\mathcal{M}}$ on the quotient $\mathcal{M}/\mathcal{I}\mathcal{M} = i^*(\mathcal{M})$. Hence, we have defined a function $\mathrm{Coh}_{\mathrm{F}}(X) \to K_\bullet(Z)$.
This fulfills (B), as $\overline{F_1 + F_2} = \overline{F_1} + \overline{F_2}$. It remains to prove that this function is additive with respect to short exact sequences. Let

$$0 \to (\mathcal{M}', F_{\mathcal{M}'}) \to (\mathcal{M}, F_{\mathcal{M}}) \to (\mathcal{M}'', F_{\mathcal{M}''}) \to 0$$

be exact in $\mathrm{Coh}_{\mathrm{F}}(X)$. Pullback of sheaves is, of course, not an exact functor so we have to look for some tricks. There is a commutative diagram, compatible with the Frobenius actions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{IM} \cap \mathcal{M}' & \longrightarrow & \mathcal{IM} & \longrightarrow & \mathcal{IM}'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{M}' & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}'' & \longrightarrow & 0. \end{array}$$

Taking quotients, we obtain a short exact sequence (e.g. by snake lemma)

$$0 \to \mathcal{M}'/\mathcal{IM} \cap \mathcal{M}' \to \mathcal{M}/\mathcal{IM} \to \mathcal{M}''/\mathcal{IM}'' \to 0.$$

By a similar argument, there is also an exact sequence

$$0 \to \mathcal{IM} \cap \mathcal{M}'/\mathcal{IM}' \to \mathcal{M}'/\mathcal{IM}' \to \mathcal{M}'/\mathcal{IM} \cap \mathcal{M}' \to 0.$$

Now, we are in good shape, because

$$\begin{aligned} [\mathcal{M}/\mathcal{IM}] &= [\mathcal{M}'/\mathcal{IM} \cap \mathcal{M}'] + [\mathcal{M}''/\mathcal{IM}''] \\ &= [\mathcal{M}'/\mathcal{IM}'] - [\mathcal{IM} \cap \mathcal{M}'/\mathcal{IM}'] + [\mathcal{M}''/\mathcal{IM}''], \end{aligned}$$

and it would suffice to prove that $[\mathcal{IM} \cap \mathcal{M}'/\mathcal{IM}'] = 0$.
If we restrict to some affine open, $\mathcal{M}$ becomes a module and the filtration $\mathcal{M} \supseteq \mathcal{IM} \supseteq \mathcal{I}^2\mathcal{M} \supseteq \ldots$ is of course $\mathcal{I}$-stable. By Artin–Rees [5, Lemma 5.1, p. 146], $\mathcal{M}' \supseteq \mathcal{IM} \cap \mathcal{M}' \supseteq \mathcal{I}^2\mathcal{M} \cap \mathcal{M}' \supseteq \ldots$ is $\mathcal{I}$-stable as well, i.e. there exists $n \geq 0$ such that $\mathcal{I}^{j+n}\mathcal{M} \cap \mathcal{M}' = \mathcal{I}^j(\mathcal{I}^n\mathcal{M} \cap \mathcal{M}')$ for all $j \geq 0$, so in particular $\mathcal{I}^k\mathcal{M} \cap \mathcal{M}' \subseteq \mathcal{IM}'$ for $k \gg 0$. As Z is Noetherian, we can take n big enough, such that $\mathcal{I}^n\mathcal{M} \cap \mathcal{M}' \subseteq \mathcal{IM}'$ as sheaves on all of Z. For large m (already $m \geq n$ suffices) this yields

$$F^m_{\mathcal{M}'}(\mathcal{IM} \cap \mathcal{M}') \subseteq \mathcal{I}^{q^m}\mathcal{M} \cap \mathcal{M}' \subseteq \mathcal{IM}'$$

and, therefore, $F_{\mathcal{M}'}$ is nilpotent on $\mathcal{IM} \cap \mathcal{M}'/\mathcal{IM}'$. We conclude by Lemma 3.9. Obviously, $i^* \circ i_* = \mathrm{id}$ for closed immersions i and so the proof is complete. □

Corollary 3.12. *If X is a scheme of finite type over $\mathbb{F}_q$ and $X_1, \ldots, X_r$ are the connected components of X, then there is an isomorphism*

$$\bigoplus_{i=1}^r K_\bullet(X_i) \simeq K_\bullet(X).$$

Proof. Let $f_i \colon X_i \hookrightarrow X$ be the inclusion. The sum of the pushforwards defines a morphism $\bigoplus_{i=1}^r K_\bullet(X_i) \to K_\bullet(X)$ with restriction of sheaves as inverse, by the previous lemma. □

Example 3.13. We consider the set of rational points of the projective space $\mathbb{P}^n$, which is a closed subscheme, equipped with the reduced structure. Using the decomposition $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$, we find that $|\mathbb{P}^n(\mathbb{F}_q)| = 1 + q + \ldots + q^n$. Then the previous result, together with Lemma 3.8 implies:

$$K_\bullet(\mathbb{P}^n(\mathbb{F}_q)) \simeq \mathbb{F}_q^{\oplus 1+q+\ldots+q^n}.$$

Lemma 3.14. *Let X be a scheme of finite type over $\mathbb{F}_q$ and $i \colon Z \hookrightarrow X$ a closed subscheme. If $(\mathcal{M}, F_{\mathcal{M}})$ is an F-module on X with support in Z, then $[\mathcal{M}, F_{\mathcal{M}}] \in i_*(K_\bullet(Z))$.*

Proof. If $\mathcal{I}$ is the ideal sheaf of Z in X, then there is a filtration $\mathcal{M} \supseteq \mathcal{I}\mathcal{M} \supseteq \ldots \supseteq \mathcal{I}^k\mathcal{M} = 0$ of F-modules, where all successive quotients are sheaves on Z. Indeed, locally $\mathcal{M}$ is a finite module M over a Noetherian ring A and I is the restriction of $\mathcal{I}$. By assumption,

$$\operatorname{Supp} M = \mathrm{V}(\operatorname{Ann} M) \subseteq \mathrm{V}(I).$$

We let $k \gg 0$ such that $I^k \subseteq \operatorname{Ann} M$, so $I^k M = 0$. As X is quasi-compact, one can take k big enough, such that $\mathcal{I}^k\mathcal{M} = 0$ globally. This yields short exact sequences, compatible with the Frobenius actions, of the form:

$$0 \to \mathcal{I}^{j+1}\mathcal{M} \to \mathcal{I}^j\mathcal{M} \to \mathcal{I}^j\mathcal{M}/\mathcal{I}^{j+1}\mathcal{M} \to 0.$$

Consequently, $[\mathcal{M}] = [\mathcal{M}/\mathcal{I}\mathcal{M}] \in K_\bullet(X)$, since the Frobenius acts trivially on $\mathcal{I}^j\mathcal{M}/\mathcal{I}^{j+1}\mathcal{M}$ for $j \geq 1$. Hence, $[\mathcal{M}, F_\mathcal{M}]$ is contained in the image of $K_\bullet(Z)$. □

Remark 3.15. For flat morphisms, the pullback of coherent sheaves is exact and respects sums of Frobenius actions. Therefore, $K_\bullet$ is contravariant for flat morphisms as well, in particular for open immersions. Combined with Lemma 3.11, we see that every immersion $f\colon X \to Y$, i.e. an isomorphism of X with an open subscheme of a closed subscheme of Y, induces a morphism:

$$\begin{aligned} K_\bullet(Y) &\to K_\bullet(X) \\ [\mathcal{M}, F_\mathcal{M}] &\mapsto [f^*(\mathcal{M}), f^*(F_\mathcal{M})]. \end{aligned}$$

3.5 Computation of $K_\bullet(\mathbb{P}^n)$

We compute the Grothendieck group $K_\bullet(\mathbb{P}^n)$, using the investigation of F-modules on the projective space from the previous section.

Lemma 3.16. *The dimension of $K_\bullet(\mathbb{P}^n)$ as an $\mathbb{F}_q$-vector space equals $1 + q + \ldots + q^n$. Moreover, the inclusion $i'\colon \mathbb{P}^n(\mathbb{F}_q) \hookrightarrow \mathbb{P}^n$ induces an isomorphism*

$$i'_*\colon K_\bullet(\mathbb{P}^n(\mathbb{F}_q)) \xrightarrow{\simeq} K_\bullet(\mathbb{P}^n).$$

The inverse is given by the restriction as described in Lemma 3.11.

In the proof we will also see that this group is generated by line bundles $\mathcal{O}(-l)$ with certain Frobenius actions and we determine an explicit class of generators.

Proof. We follow the proof in [18, Prop. 5.13, pp. 45–47].

We proceed in several steps:

Step 1: Suppose that $\mathcal{M}$ is a coherent sheaf on $\mathbb{P}^n$ with a Frobenius action $F_\mathcal{M}$. From the discussion in Section 2 we know that we can pass to its associated graded S-module M, together with an endomorphism φ.

There exists a graded free resolution of M by the Hilbert Syzygy Theorem (see e.g. [5, 19.7]),

$$0 \to F_{n+1} \to \ldots \to F_1 \to F_0 \to M \to 0,$$

such that $F_j = \bigoplus_i S(-k_{i,j})$ with $k_{i,j} \in \mathbb{Z}$.

As F_0 is projective, the endomorphism φ of M can be lifted to give a graded Frobenius action φ_0 on F_0. Next, we consider the kernel K of $F_0 \to M$ and observe that F_1 surjects onto K. We use that F_1 is projective as well to lift φ_0 and then we continue in this manner. Eventually, we get an exact sequence that is compatible with the graded Frobenius actions, which yields a relation (by Lemma 3.9 (iv)) in $K_\bullet(\mathbb{P}^n)$:

$$[\widetilde{M}] = \sum_{i=0}^{n+1} [\widetilde{F}_i].$$

Now Lemma 3.9 (ii) implies that $K_\bullet(\mathbb{P}^n)$ is generated, as an $\mathbb{F}_q$-vector space, by the set of all $[\mathcal{O}(-l), x_0^{a_0} \cdots x_n^{a_n}]$, where $a_i \geq 0$ and $\sum_{i=0}^{n} a_i = ql - l$.

Step 2: We detect a crucial relation that holds in $K_\bullet(\mathbb{P}^n)$ to eliminate some of the generators that we determined in the first step.
For this, consider the hyperplane $H = V(x_i)$ in $\mathbb{P}^n$ that comes with a short exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_H \to 0.$$

We twist by $\mathcal{O}(-l+1)$ and notice that the resulting sequence is compatible with the Frobenius actions given by $x_i^q f$ resp. $x_i f$ with $f \in H^0(\mathbb{P}^n, \mathcal{O}(ql-l-q))$ (cf. Example 2.19):

$$0 \to (\mathcal{O}(-l), x_i^q f) \to (\mathcal{O}(-l+1), x_i f) \to (\mathcal{O}_H(-l+1), 0) \to 0.$$

We use that $\mathcal{O}_H = \widetilde{S/(x_i)}$, so the induced graded Frobenius action by $x_i f$ on the quotient is zero and its class in the Grothendieck group is zero as well by Remark 3.6. We deduce that

$$[\mathcal{O}(-l), x_i^q f] = [\mathcal{O}(-l+1), x_i f] \in K_\bullet(\mathbb{P}^n).$$

This allows us to reduce the degree of a monomial that determines a Frobenius action, whenever there is some exponent greater than $q-1$.

Step 3: So far, we have shown that it suffices to consider only those monomials $x_0^{a_0} \cdots x_n^{a_n}$ with $a_i \leq q-1$ for all i. In a final step, we will prove that also $[\mathcal{O}(-n-1), x_0^{q-1} \cdots x_n^{q-1}]$ can be removed from our set of generators. For this purpose, we equip $S(-1)$ with the action given by x_i^{q-1} and denote the associated F-module on $\mathbb{P}^n$ by $\mathcal{L}_i$. We obtain a complex of graded Frobenius modules (cf. Example 2.19):

$$0 \to S(-1) \xrightarrow{\cdot x_i} S \to 0.$$

The tensor product of all these complexes yields the Koszul complex, associated to the regular sequence $x_0, \ldots, x_n \in S$:

$$0 \to E_{n+1} \to E_n \to \ldots \to E_1 = S(-1)^{\oplus n+1} \to S \to 0.$$

The cokernel at the right end is $C = S/(x_0, \ldots, x_n)$. But C as a graded S-module is concentrated in a finite number of degrees (actually just in degree zero). Hence $\widetilde{C} = 0$ and the sequence becomes exact after passing to the associated sheaves on $\mathbb{P}^n$:

$$0 \to \mathcal{E}_{n+1} \to \ldots \to \mathcal{E}_1 = \mathcal{L}_0 \oplus \ldots \oplus \mathcal{L}_n \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$

Then

$$\mathcal{E}_r = \bigoplus_{0 \leq i_1 < \ldots < i_r \leq n} (\mathcal{L}_{i_1} \otimes \ldots \otimes \mathcal{L}_{i_r}) \simeq \mathcal{O}(-r)^{\oplus \binom{n+1}{r}}$$

and the action on $\mathcal{O}(-r) \simeq \mathcal{L}_{i_1} \otimes \ldots \otimes \mathcal{L}_{i_r}$ is given by $x_{i_1}^{q-1} \cdots x_{i_r}^{q-1}$. By Lemma 3.9, we can express $[\mathcal{O}(-n-1), x_0^{q-1} \cdots x_n^{q-1}]$ in terms of the remaining generators.
Step 4: We put everything together and count how many generators are left. This number will be called b_n.

The condition that the monomial $x_0^{a_0} \cdots x_n^{a_n}$ defines an action on $\mathcal{O}(-l)$ for some $l \geq 0$ ensures that $\sum_{i=0}^{n} a_i = (q-1)l$, thus $\sum_{i=0}^{n} a_i \equiv 0 \mod q-1$. We compute b_n by induction, distinguishing two cases:

Assume we are given $a_0, \ldots, a_{n-1}$, satisfying $0 \leq a_i \leq q-1$ with at least one $a_j < q-1$, such that $\sum_{i=0}^{n-1} a_i \equiv 0 \mod q-1$. There are b_{n-1} such tuples $a_0, \ldots, a_{n-1}$. Then a_n can either be 0 or $q-1$.

Otherwise, if $a_0 = \ldots = a_{n-1} = q-1$ or if $\sum_{i=0}^{n-1} a_i \not\equiv 0 \mod q-1$ and $0 \leq a_i \leq q-1$, which happens for $q^n - b_{n-1}$ tuples, there is precisely one choice for a_n to make $x_0^{a_0} \cdots x_n^{a_n}$ a valid monomial in the sense of the discussion above. We conclude that

$$b_n = 2 \cdot b_{n-1} + 1 \cdot (q^n - b_{n-1}) = b_{n-1} + q^n.$$

As $b_0 = 1$, it follows that $\dim_{\mathbb{F}_q} K_\bullet(\mathbb{P}^n) \leq 1 + q + \ldots + q^n$.

The number of rational points of the projective space is $|\mathbb{P}^n(\mathbb{F}_q)| = 1 + q + \ldots + q^n$ and by Corollary 3.12 this is precisely the dimension of $K_\bullet(\mathbb{P}^n(\mathbb{F}_q))$ as an $\mathbb{F}_q$-vector space. The inclusion $i' \colon \mathbb{P}^n(\mathbb{F}_q) \hookrightarrow \mathbb{P}^n$ induces an injection $K_\bullet(\mathbb{P}^n(\mathbb{F}_q)) \xrightarrow{i'_*} K_\bullet(\mathbb{P}^n)$ by Lemma 3.11. Comparing dimensions, this has to be an isomorphism and the proof is complete. □

Corollary 3.17. *For the projective space, the Poincaré homomorphism*

$$c_{\mathbb{P}^n} \colon K^\bullet(\mathbb{P}^n) \xrightarrow{\simeq} K_\bullet(\mathbb{P}^n)$$

is an isomorphism.

Proof. Surjectivity follows from the proof of the previous lemma, since $K_\bullet(\mathbb{P}^n)$ is generated by line bundles. We could have done the same discussion just with locally free sheaves and conclude, that $K^\bullet(\mathbb{P}^n)$ is generated by the same line bundles and so the assertion follows. □

4 The main theorem

In this section we give a proof of the localization theorem, from which we deduce Fulton's trace formula.

Theorem 4.1 (Localization Theorem)**.** *Let X be a scheme of finite type over $\mathbb{F}_q$. Then the inclusion $i \colon X(\mathbb{F}_q) \hookrightarrow X$ induces an isomorphism*

$$i_* \colon K_\bullet(X(\mathbb{F}_q)) \xrightarrow{\simeq} K_\bullet(X).$$

The inverse is given by the restriction i^* as described in Lemma 3.11.

Proof. We follow the proof in [7, pp. 193–194].

The proof will consist of several steps. Using the computation of $K_\bullet(\mathbb{P}^n)$, we have already seen that the theorem holds true for the projective space. The case of an arbitrary projective scheme will then be a formal consequence. Afterwards, we treat the case of a quasi-projective scheme and finally we use

Chow's Lemma to prove the assertion for X of finite type over $\mathbb{F}_q$.

Step 1: For a projective scheme X over $\mathbb{F}_q$ we fix a closed immersion $j\colon X \hookrightarrow \mathbb{P}^n$. The restriction of j to the set of rational points is called j' and we let i be the inclusion $X(\mathbb{F}_q) \hookrightarrow X$. There is a commutative diagram

$$\begin{array}{ccc} X(\mathbb{F}_q) & \xhookrightarrow{j'} & \mathbb{P}^n(\mathbb{F}_q) \\ \big\downarrow i & & \big\downarrow i' \\ X & \xhookrightarrow{j} & \mathbb{P}^n. \end{array}$$

We will make use of the fact that $i'^* \circ j_* = j'_* \circ i^*$ which follows from the observation that $X \cap \mathbb{P}^n(\mathbb{F}_q) = X(\mathbb{F}_q)$ as closed subschemes of $\mathbb{P}^n$. Now, we apply $K_\bullet$ and use Lemma 3.16:

$$j_* i_* i^* = i'_* j'_* i^* = i'_* i'^* j_* = j_*.$$

Composing with j^* on the left gives the desired identity $i_* \circ i^* = \mathrm{id}_{K_\bullet(X)}$. By Lemma 3.11 the assertion follows:

$$i_*\colon K_\bullet(X(\mathbb{F}_q)) \xrightarrow{\simeq} K_\bullet(X),$$

with inverse i^*.

Step 2: Next, we consider the case when X is quasi-projective.

As the theorem holds true for $\mathbb{P}^n$, we have that $[\mathcal{O}_{\mathbb{P}^n}, \mathrm{Frob_q}] = [i_*\mathcal{O}_{\mathbb{P}^n(\mathbb{F}_q)}, \mathrm{id}] \in K_\bullet(\mathbb{P}^n)$, since the Frobenius on $\mathbb{P}^n(\mathbb{F}_q)$ is the identity. The pushforward $i_*\mathcal{O}_{\mathbb{P}^n(\mathbb{F}_q)}$ is a coherent F-module and we use Step 1 of the proof of Lemma 3.16 to obtain a finite resolution by locally free F-modules:

$$0 \to \mathcal{F}_{n+1} \to \ldots \to \mathcal{F}_0 \to i_*\mathcal{O}_{\mathbb{P}^n(\mathbb{F}_q)} \to 0.$$

Thus, $[\mathcal{O}_{\mathbb{P}^n}] = \sum_{i=0}^{n+1} (-1)^i [\mathcal{F}_i] \in K_\bullet(\mathbb{P}^n)$ by Lemma 3.9 and this identity holds in $K^\bullet(\mathbb{P}^n)$ as well, using the isomorphism from Corollary 3.17.

If $f\colon X \to \mathbb{P}^n$ is an immersion, we obtain that $[\mathcal{O}_X, \mathrm{Frob_q}] = \sum_{i=0}^{n+1} (-1)^i [f^*(\mathcal{F}_i)] \in K^\bullet(X)$. We make use of the module structure of $K_\bullet(X)$ over $K^\bullet(X)$ and find that

$$[\mathcal{M}] = [\mathcal{O}_X \otimes \mathcal{M}] = \sum_{i=0}^{n+1} (-1)^i [f^*(\mathcal{F}_i) \otimes \mathcal{M}],$$

for any F-module $\mathcal{M}$ on X.

By Lemma 3.9, we can express this as an alternating sum of the homology sheaves of the complex $f^*(\mathcal{F}_\bullet) \otimes \mathcal{M}$, i.e. $[\mathcal{M}] = \sum_{i=0}^{n+1} (-1)^i [\mathcal{H}_i]$. Observe that for each point $x \in \mathbb{P}^n \smallsetminus \mathbb{P}^n(\mathbb{F}_q)$ there is a neighbourhood where $\mathcal{F}_\bullet$ is exact. Therefore, the pullback to X and also tensoring with $\mathcal{M}$ stays exact, since all $\mathcal{F}_i$ are locally free. Thus, the homology sheaves are supported on $f^{-1}(\mathbb{P}^n(\mathbb{F}_q)) = X(\mathbb{F}_q)$. Using Lemma 3.14 we see that $K_\bullet(X(\mathbb{F}_q)) \to K_\bullet(X)$ is indeed surjective.

Step 3: For general X we use Chow's Lemma ([23, Lemma 29.18.1]) and obtain a proper morphism $f\colon \widetilde{X} \to X$ over $\mathbb{F}_q$ with $\widetilde{X}$ quasi-projective. In fact, f can be chosen such that it induces an isomorphism $f^{-1}(U) \to U$ for $U \subseteq X$ open, dense.

Let $(\mathcal{M}, F_{\mathcal{M}})$ be an F-module on X. The unit $\mathrm{id} \to f_* \circ f^*$ induces an exact sequence of F-modules (we plugged in the kernel and cokernel):

$$0 \to \mathcal{K} \to \mathcal{M} \to f_* f^*(\mathcal{M}) \to \mathcal{C} \to 0.$$

On U, the pullback and pushforward are inverse to each other, hence $\mathcal{C}$ and $\mathcal{K}$ are supported on $Y = X \smallsetminus U$. By the same argument, the pushforward f_* is exact on $f^{-1}(U)$ and all higher direct images of $\mathcal{O}_{\widetilde{X}}$-modules are supported on Y. Then the following equality holds in $K_\bullet(X)$:

$$\begin{aligned}[\mathcal{M}] &= [\mathcal{K}] - [\mathcal{C}] + [f_* f^*(\mathcal{M})] \\ &= [\mathcal{K}] - [\mathcal{C}] + f_*[f^*(\mathcal{M})] - \sum_{i \geq 1} (-1)^i [R^i f_* f^*(\mathcal{M})].\end{aligned}$$

Consequently, the morphism $K_\bullet(\widetilde{X}) \oplus K_\bullet(Y) \to K_\bullet(X)$ is surjective.
By noetherian induction, we may assume that the theorem holds true for Y. Since $\widetilde{X}$ is quasi-projective, we can use Step 2 and obtain the following commutative diagram:

$$\begin{array}{ccc} K_\bullet(\widetilde{X}(\mathbb{F}_q)) \oplus K_\bullet(Y(\mathbb{F}_q)) & \longrightarrow & K_\bullet(X(\mathbb{F}_q)) \\ \downarrow \wr & & \downarrow i_* \\ K_\bullet(\widetilde{X}) \oplus K_\bullet(Y) & \twoheadrightarrow & K_\bullet(X). \end{array}$$

Then it follows that i_* is surjective as well and we are done. $\square$

Corollary 4.2. *Let X be a scheme of finite type over $\mathbb{F}_q$ and $i\colon Z \hookrightarrow X$ a closed subscheme. Denote by U the open subscheme $X \smallsetminus Z$ and $j\colon U \hookrightarrow X$ the inclusion. Then there is an exact sequence*

$$0 \longrightarrow K_\bullet(Z) \xrightarrow{i_*} K_\bullet(X) \xrightarrow{j^*} K_\bullet(U) \longrightarrow 0.$$

Proof. We consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_\bullet(Z(\mathbb{F}_q)) & \xrightarrow{i_*} & K_\bullet(X(\mathbb{F}_q)) & \xrightarrow{j^*} & K_\bullet(U(\mathbb{F}_q)) & \longrightarrow & 0 \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\ 0 & \longrightarrow & K_\bullet(Z) & \xrightarrow{i_*} & K_\bullet(X) & \xrightarrow{j^*} & K_\bullet(U) & \longrightarrow & 0. \end{array}$$

The upper sequence is exact by Corollary 3.12 and the squares commute, since the isomorphisms are given by pushforwards. Hence the lower sequence is exact as well. $\square$

Theorem 4.3 (Fixed Point Formula)**.** *Let X be a proper scheme of dimension d over $\mathbb{F}_q$ and let $(\mathcal{M}, F_{\mathcal{M}})$ be a coherent F-module on X. Then*

$$\sum_{x \in X(\mathbb{F}_q)} \mathrm{trace}(F_{\mathcal{M}}(x)) = \sum_{i=0}^{d} (-1)^i \, \mathrm{trace}(F_{\mathcal{M}} | H^i(X, \mathcal{M})) \in \mathbb{F}_q. \tag{1}$$

Proof. Due to Theorem 4.1, there is a commutative diagram:

$$\begin{array}{ccc} K_\bullet(X(\mathbb{F}_q)) & \xleftarrow{\quad i^* \quad} & K_\bullet(X) \\ & {\scriptstyle p_*} \searrow \quad \swarrow {\scriptstyle \pi_*} & \\ & K_\bullet(\operatorname{Spec} \mathbb{F}_q), & \end{array}$$

where p and π are the structure morphisms. Then the higher direct image $R^i\pi_*$ is the same as cohomology H^i, therefore

$$\pi_*[\mathcal{M}, F_{\mathcal{M}}] = \sum_{i=0}^{d}(-1)^i[H^i(X,\mathcal{M}), F_{\mathcal{M}}].$$

Implicitly, we make use of the fact that $K_\bullet(X(\mathbb{F}_q)) \simeq \bigoplus_{x\in X(\mathbb{F}_q)} K_\bullet(\operatorname{Spec}\mathbb{F}_q)$ from Lemma 3.12. Then the pullback of $\mathcal{M}$ is the collection of all the fibres over rational points:

$$i^*[\mathcal{M}, F_{\mathcal{M}}] = ([\mathcal{M}_x \otimes k(x), F_{\mathcal{M}}(x)])_{x\in X(\mathbb{F}_q)}.$$

Again, R^ip_* corresponds to taking the i^{th} cohomology, but as $X(\mathbb{F}_q)$ is zero-dimensional, the pushforward on Grothendieck groups degenerates to:

$$p_* \circ i^*[\mathcal{M}, F_{\mathcal{M}}] = \sum_{x\in X(\mathbb{F}_q)} [\mathcal{M}_x \otimes k(x), F_{\mathcal{M}}(x)].$$

Finally, we use Lemma 3.8 which gives an isomorphism of $K_\bullet(\operatorname{Spec}\mathbb{F}_q)$ with $\mathbb{F}_q$ via the trace of the Frobenius action. □

Remark 4.4. The assumption that X is proper over $\mathbb{F}_q$ is crucial. It ensures for instance that the cohomology groups are finite-dimensional as $\mathbb{F}_q$- vector spaces. Already the pushforward $K_\bullet(X) \to K_\bullet(\operatorname{Spec}\mathbb{F}_q)$ does not even exist if we drop the condition of properness (consider e.g. the affine line).

The trace formula, announced in the introduction, is now a simple consequence.

Theorem 4.5 (Trace Formula). *If X is a proper scheme of dimension d over a finite field $\mathbb{F}_q$, then*

$$|X(\mathbb{F}_q)| \equiv \sum_{i=0}^{d}(-1)^i \operatorname{trace}(\mathrm{Frob_q}\,|H^i(X,\mathcal{O}_X)) \mod p. \tag{2}$$

Proof. Consider the F-module $(\mathcal{O}_X, \mathrm{Frob_q})$ and bear in mind that the fibres over rational points are equal to $\mathbb{F}_q$ and so the Frobenius acts as the identity on all of these. What remains on the left hand side of the fixed point formula (1) is the number of rational points. □

In particular, this implies that the right hand side is actually contained in $\mathbb{F}_p$, which is a priori not clear at all. The result is interesting, because it is useful in both directions. Not only that we can calculate $|X(\mathbb{F}_q)|$ modulo p, but if we already know the number of rational points, say by using a computer, we get information about cohomology.

Using a base change to a finite extension $\mathbb{F}_q \hookrightarrow \mathbb{F}_{q^m}$ we obtain the following result:

Corollary 4.6. *If X is a proper scheme of dimension d over a finite field $\mathbb{F}_q$, then*

$$|X(\mathbb{F}_{q^m})| \equiv \sum_{i=0}^{d}(-1)^i \operatorname{trace}(\mathrm{Frob_q^m}\,|H^i(X,\mathcal{O}_X)) \mod p.$$

Proof. Observe that the Frobenius on the base change $X_m = X \times_{\mathbb{F}_q} \mathbb{F}_{q^m}$ equals $\mathrm{Frob}_q^m \times \mathrm{id}$ and we have a canonical isomorphism

$$H^i(X_m, \mathcal{O}_{X_m}) \simeq H^i(X, \mathcal{O}_X) \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}.$$

The assertion now follows immediately from the trace formula (2). □

Remark 4.7. Passing from X to its reduction X_{red} does not change the number of rational points, but it is far from clear that all information, encoded by the right hand side of the formula, is captured by the reduced scheme structure of X.

5 Examples and applications

The results of the previous sections will be used to study the number of rational points on certain schemes. We will focus on the case of Calabi–Yau varieties.

5.1 Chevalley–Warning and p-adic estimates

We obtain the Chevalley–Warning theorem (which can be found in [4], [24]) as a consequence of the trace formula. This classical result can be stated in two versions that we quickly prove to be equivalent.

Theorem 5.1. *Let $\mathbb{F}_q$ be a finite field of characteristic p. Then the following holds:*

(i) *If $F_1, \dots, F_r \in \mathbb{F}_q[x_1, \dots, x_n]$ are polynomials of degree $\deg(F_i) = d_i$ with $d_1 + \ldots + d_r < n$ defining an affine variety X, then:*

$$|X(\mathbb{F}_q)| \equiv 0 \mod p.$$

(ii) *If $F_1, \dots, F_r \in \mathbb{F}_q[x_0, \dots, x_n]$ are homogeneous polynomials of degree $\deg(F_i) = d_i$ with $d_1 + \ldots + d_r < n + 1$ defining a projective variety X, then:*

$$|X(\mathbb{F}_q)| \equiv 1 \mod p.$$

Remark 5.2. Henceforth we also use the shorthand $N(X) = |X(\mathbb{F}_q)|$.
To show that the two assertions are equivalent, we first assume that (i) holds true. Denote by C the affine cone of the projective variety X. Then $N(C)$ is divisible by p and:

$$N(X) = \frac{N(C) - 1}{q - 1}.$$

This implies that

$$N(X) - 1 = \frac{N(C) - q}{q - 1},$$

which is divisible by p as well and we get $N(X) \equiv 1 \mod p$.
For the other implication we let X be the affine variety given by $F_1, \dots, F_r \in \mathbb{F}_q[x_1, \dots, x_n]$ with $\sum_{i=1}^{r} d_i < n$. Introducing a variable x_0 gives rise to a projective variety $Y \subseteq \mathbb{P}^n$, such that $Y \cap \{x_0 = 1\} = X$. We consider $Y \cap \{x_0 = 0\}$, which is a projective variety in $\mathbb{P}^{n-1}$. The restriction of the polynomials still satisfies $\sum_{i=1}^{r} d_i < n$ so we can apply (ii) to Y and $Y \cap \{x_0 = 0\}$ which yields:

$$N(X) \equiv N(Y) - N(Y \cap \{x_0 = 0\}) \equiv 1 - 1 \equiv 0 \mod p.$$

Proof of 5.1. We prove the second version of the theorem.

The condition $d_1 + \ldots + d_r < n + 1$ ensures that the structure sheaf $\mathcal{O}_X$ has no higher cohomology (by a standard argument which can be found e.g. in [22, Prop. 4.3, pp. 58–59]). We treat the case when X is a complete intersection given by $F_1, \ldots, F_r$.

Consider the locally free sheaf $\mathcal{E} = \bigoplus_i \mathcal{O}(d_i)$ and the global section $(F_1, \ldots, F_r)$. The associated Koszul complex is a finite locally free resolution of $\mathcal{O}_X$ (see e.g. [8, pp. 75–77] or [5, III §17 pp. 427–436] for a more detailed discussion of Koszul complexes):

$$0 \to \Lambda^r \mathcal{E}^* \to \Lambda^{r-1} \mathcal{E}^* \to \ldots \to \Lambda^2 \mathcal{E}^* \to \mathcal{E}^* \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0.$$

From this, we extract that $\mathcal{O}_X$ has no higher cohomology since $\mathcal{O}(l)$ is acyclic for $l > -n-1$. The Frobenius acts as the identity on global sections so the assertion follows immediately from the trace formula. □

Remark 5.3. In contrast to the above argument, the original proof of the Chevalley–Warning theorem is very short, using an ingenious polynomial argument.
That this theorem follows from the trace formula should not be seen as a new proof but rather as a sign that coherent cohomology is fine enough to detect congruences modulo p.

We state a generalization which is due to Ax and Katz.

Proposition 5.4. *Let $X \subseteq \mathbb{P}^n$ be the projective variety given by homogeneous polynomials $F_1, \ldots, F_r \in \mathbb{F}_q[x_0, \ldots, x_n]$ of degree $\deg(F_i) = d_i$. Assume that $d_1 \geq \ldots \geq d_r$ and $\sum_{i=1}^{r} d_i < n + 1$. Then for each positive integer b, satisfying $b < \frac{n+1-(d_1+\ldots+d_r)}{d_1} + 1$, it holds that:*

$$|X(\mathbb{F}_q)| \equiv |\mathbb{P}^n(\mathbb{F}_q)| \mod q^b.$$

Proof. A proof can be found in [1] and [14]. □

By taking $b = 1$, we obtain a congruence modulo q. A new proof of this result was later given by Berthelot, Bloch and Esnault, using Witt vector cohomology (cf. [2]).

5.2 Applications to Calabi–Yau varieties and further examples

We observe that for Calabi–Yau varieties the notion of Frobenius split is closely related to the existence of a rational point. Later we investigate two classes of Calabi–Yau hypersurfaces in the projective space and distinguish them, using the trace formula.

Definition 5.5. *Let X be a smooth, proper variety of dimension n and ω_X its canonical bundle. We say that X is a Calabi–Yau variety, if*

(i) $\omega_X \simeq \mathcal{O}_X$ *and*

(i) $H^i(X, \mathcal{O}_X) = 0$ *for* $0 < i < n$.

Proposition 5.6. *If X is a Calabi–Yau variety over $\mathbb{F}_q$, then X is Frobenius split if and only if $|X(\mathbb{F}_q)| \not\equiv 1 \mod p$.*

Proof. This is immediate from Lemma 2.12, together with the trace formula. □

Remark 5.7. The previous lemma in particular implies that a Calabi–Yau variety has a rational point if it is not split. To check, whether a variety X is not split can be relatively easy; according to [3, Thm. 1.2.8, p. 16], it would suffice to find an ample line bundle $\mathcal{L} \in \operatorname{Pic}(X)$ and some $i \geq 1$ with $H^i(X, \mathcal{L}) \neq 0$.

Proposition 5.8. *Let $f \in \mathbb{F}_q[x_0, \dots, x_n]$ be a homogeneous polynomial of degree $n+1$, with $n \geq 2$, defining the hypersurface $X \subseteq \mathbb{P}^n$. The following assertions are equivalent:*

(i) *The action induced by the Frobenius morphism on $H^{n-1}(X, \mathcal{O}_X)$ is bijective (equivalently, it is nonzero).*

(ii) $|X(\mathbb{F}_q)| \not\equiv 1 \mod p$.

(iii) *The coefficient of $(x_0 \cdots x_n)^{q-1}$ in f^{q-1} is nonzero.*

(iv) *The coefficient of $(x_0 \cdots x_n)^{p-1}$ in f^{p-1} is nonzero.*

Proof. We follow the proof in [18, Prop. 5.15, pp. 47–48].

The ideal sheaf of X in $\mathbb{P}^n$ is $\mathcal{O}(-n-1)$, so there is the short exact sequence

$$0 \to \mathcal{O}(-n-1) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0.$$

Thus, $\mathcal{O}_X$ has cohomology only in degree 0 and $n-1$ namely $h^0(\mathcal{O}_X) = h^{n-1}(\mathcal{O}_X) = 1$. The Frobenius on the global sections $H^0(X, \mathcal{O}_X) \simeq \mathbb{F}_q$ is the identity, so the trace formula looks like:

$$|X(\mathbb{F}_q)| \equiv 1 + (-1)^{n-1} \operatorname{trace}(\mathrm{Frob_q} | H^{n-1}(X, \mathcal{O}_X)) \mod p.$$

As $h^{n-1}(\mathcal{O}_X) = 1$, the trace vanishes if and only if the action on $H^{n-1}(X, \mathcal{O}_X)$ vanishes and we have proved the equivalence of (i) and (ii).

Let c_r be the coefficient of $(x_0 \cdots x_n)^{p^r-1}$ in f^{p^r-1}. By an elementary calculation (cf. [18, p. 48]) we find that $c_r = c_1^{1+p+\ldots+p^{r-1}}$, which proves that (iii) and (iv) are equivalent.

The short exact sequence mentioned above gives rise to a short exact sequence of Čech complexes with respect to the standard affine open cover of $\mathbb{P}^n$. We keep in mind that $H^n(\mathbb{P}^n, \mathcal{O}(-n-1))$ is generated by $\frac{1}{x_0 \cdots x_n}$ as an $\mathbb{F}_q$- vector space (see e.g [11, Thm. 5.1, pp. 225–228]). Computing the boundary operator $\delta \colon H^{n-1}(X, \mathcal{O}_X) \xrightarrow{\simeq} H^n(\mathbb{P}^n, \mathcal{O}(-n-1))$ explicitly shows that the Frobenius on the cohomology of X corresponds to the endomorphism $w \mapsto f^{q-1}w^q$ of $H^n(\mathbb{P}^n, \mathcal{O}(-n-1))$. We express f^{q-1} uniquely as $f^{q-1} = c_r(x_0 \cdots x_n)^{q-1} + g$ with $g \in (x_0^q, \dots, x_n^q)$. Then

$$\frac{1}{x_0 \cdots x_n} \mapsto (c_r(x_0 \cdots x_n)^{q-1} + g) \left(\frac{1}{x_0 \cdots x_n} \right)^q = c_r \cdot \frac{1}{x_0 \cdots x_n},$$

since all other monomials become zero. □

Remark 5.9. If in addition X is smooth and satisfies one (and hence all) of the above conditions, X is called 'ordinary'. Otherwise, it is called 'supersingular'.

Example 5.10. The proposition proposes a class of examples where the theorem of Chevalley–Warning fails, as the degrees of the polynomials are too big. As an explicit one we consider the Fermat hypersurface X defined by $f = x_0^{n+1} + \ldots + x_n^{n+1}$ in $\mathbb{P}^n$. Furthermore, we assume that the characteristic of $\mathbb{F}_q$ is $p = (n+1)l + 1$ for some $l \in \mathbb{Z}$. Then X is smooth as the Jacobian vanishes nowhere and using some combinatorics we compute that the coefficient of $(x_0 \cdots x_n)^{p-1}$ in f^{p-1} is

$$c_1 = \frac{(p-1)!}{l!^{n+1}}.$$

This is clearly not divisible by p. Thus, by Proposition 5.8, X is ordinary and the congruence suggested by Chevalley–Warning does not hold.

Example 5.11. We consider an elliptic curve $C \subseteq \mathbb{P}^2$ over $\mathbb{F}_p$ with $p \geq 5$. By the Hasse bound ([12]), there is always the estimate $|N(C) - (p+1)| \leq 2\sqrt{p}$. If we assume that C is supersingular, i.e. $N(C) \equiv 1 \mod p$ by Proposition 5.8, then actually $N(C) = p+1$ as $2\sqrt{p} < p$. Thus:

$$\text{C is supersingular} \iff |C(\mathbb{F}_p)| = p + 1.$$

Example 5.12. We can extend our result to complete intersections X in $\mathbb{P}^n$ of multidegree $(d_1, \ldots, d_r)$ with $\sum_i d_i = n+1$. For instance, if we consider the scheme $X = X_1 \cap X_2$ of codimension two, defined by F_1 and F_2 with $\deg(F_1) + \deg(F_2) = n+1$ and denote by $Y = X_1 \cup X_2 = V(F_1 \cdot F_2)$ the union of the two hypersurfaces, we find that $\mathcal{O}_{X_i}$ has no higher cohomology and hence $N(X_i) \equiv 1 \mod p$ by the trace formula.

The number of rational points of Y can be determined as the sum $N(X_1) + N(X_2)$ less those points that we counted twice, namely $N(X)$ and we get:

$$N(X) \equiv N(X_1) + N(X_2) - N(Y) \equiv 2 - N(Y) \mod p.$$

By Proposition 5.8 we conclude that

$$\begin{aligned} |X(\mathbb{F}_q)| \equiv 1 \mod p &\iff |Y(\mathbb{F}_q)| \equiv 1 \mod p \\ &\iff 0 = c_r = \text{coeff. of } (x_0 \cdots x_n)^{q-1} \text{ in } (F_1 \cdot F_2)^{q-1} \\ &\iff 0 = c_1 = \text{coeff. of } (x_0 \cdots x_n)^{p-1} \text{ in } (F_1 \cdot F_2)^{p-1}. \end{aligned}$$

Proposition 5.13. *Let $X \subseteq \mathbb{P}^n$ be a complete intersection of codimension r, defined by homogeneous polynomials $F_1, \ldots, F_r$ of degree $\deg(F_i) = d_i$. Furthermore, we assume that $d_1 + \ldots + d_r = n+1$. Then the following assertions are equivalent:*

(i) *The action induced by the Frobenius morphism on the cohomology group $H^{n-r}(X, \mathcal{O}_X)$ is bijective (equivalently, it is nonzero).*

(ii) $|X(\mathbb{F}_q)| \not\equiv 1 \mod p$.

(iii) *The hypersurface defined by $(F_1 \cdots F_r)$ is ordinary.*

Proof. A similar argument as in the proof of Corollary 5.1 shows that $\mathcal{O}_X$ has cohomology only in degree 0 and $n-r$, namely $h^0(\mathcal{O}_X) = h^{n-r}(\mathcal{O}_X) = 1$. This proves the equivalence of (i) and (ii).

We denote by X_i the hypersurface defined by F_i, so that $X = X_1 \cap \ldots \cap X_r$ and let $Y = X_1 \cup \ldots \cup X_r = V(F_1 \cdots F_r)$.

To show the equivalence with (iii), we use the formula of the previous example for the number of rational points and find by induction that:

$$N(X_1 \cap \ldots \cap X_r) = \sum_{i_1} N(X_{i_1}) - \sum_{i_1<i_2} N(X_{i_1} \cup X_{i_2}) \\ + \sum_{i_1<i_2<i_3} N(X_{i_1} \cup X_{i_2} \cup X_{i_3}) - \ldots + (-1)^{r+1} N(X_1 \cup \ldots \cup X_r). \qquad (\star)$$

Applying the trace formula we obtain that $N(X_{i_1} \cup \ldots \cup X_{i_k}) \equiv 1 \mod p$, whenever $0 \leq k < r$, since then $X_{i_1} \cup \ldots \cup X_{i_k}$ is a hypersurface defined by a polynomial of degree at most n and so the structure sheaf has no higher cohomology. Together with $(\star)$ this yields:

$$\begin{aligned} N(X) &\equiv \binom{r}{1} - \binom{r}{2} + \binom{r}{3} - \ldots + (-1)^r \binom{r}{r-1} + (-1)^{r+1} N(Y) \mod p \\ &\equiv (-1) \cdot \left(\left(\sum_{i=0}^{r} (-1)^i \binom{r}{i} \right) - \binom{r}{0} - (-1)^r \binom{r}{r} \right) + (-1)^{r+1} N(Y) \mod p \\ &\equiv (-1) \cdot (1-1)^r + 1 + (-1)^r + (-1)^{r+1} N(Y) \mod p \\ &\equiv 1 + (-1)^r + (-1)^{r+1} N(Y) \mod p. \end{aligned}$$

Then

$$N(X) \equiv \begin{cases} N(Y) \mod p & \text{r odd} \\ 2 - N(Y) \mod p & \text{r even} \end{cases}$$

and we conclude independence of r:

$$\begin{aligned} |X(\mathbb{F}_q)| \equiv 1 \mod p &\iff |Y(\mathbb{F}_q)| \equiv 1 \mod p \\ &\iff \text{Y is supersingular hypersurface.} \end{aligned}$$ □

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